BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

Volume VI Numéro 5

VARSOVIE 1958

Le Bulletin de l'Académie Polonaise des Sciences est publié en Séries.

La présente Série comprend des communications du domaine de la mathématique, astronomie et physique.

Lesdites sciences sont du domaine de la Troisième Classe de l'Académie Polonaise des Sciences.

La présente Série paraît mensuellement.

Les communications destinées à être publiées dans le Bulletin sont présentées à la Rédaction du Bulletin par les membres de l'Académie

Adresse de la Rédaction: Pałac Kultury i Nauki, Varsovie

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PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE – DZIAŁ CZASOPISM
Warszawa, Krakowskie Przedmieście 79

Nakład 947+103 egz.	*Rękopis dostarczono 3. IV. 1958*
Ark. wyd. 4,75, ark. druk. 5 3/8	*Podpisano do druku 24. VI. 1958*
Papier bezdrzewny sat. 80 g kl. III	*Druk ukończono w lipcu 1958*
Format B 5, 70×100 cm	*Zam. prod. 254/58 Cena zł 20.–*

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. —Vol. VI, No. 5. 1958

MATHEMATICS

Some Theorems Concerning Determinants

by
S. GLIDMAN

Presented by A. MOSTOWSKI on February 22, 1958

Let $\bar{a}\cdot\bar{b}$ be a scalar product of vectors $\bar{a}$ and $\bar{b}$ belonging to the same linear n-dimensional space X_n over the field of real numbers. The vector $\bar{w}$ with components

$$w_i = a_k b_l - a_l b_k\,, \qquad k<l, \qquad i = 1, 2, \ldots, \frac{n(n-1)}{2}$$

is called the external product of $\bar{a}$ and $\bar{b}$. This product will be denoted by

$$\bar{w} = \bar{a}\times\bar{b}\,. \tag{1}$$

1. It is easy to see, that for any $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \epsilon\, X_n$ we have

$$(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d}) - (\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c}) = (\bar{a}\times\bar{b})(\bar{c}\times\bar{d})\,. \tag{2}$$

Indeed,

$$\sum_{k=1}^{n} a_k c_k \sum_{l=1}^{n} b_l d_l - \sum_{k=1}^{n} b_k c_k \sum_{l=1}^{n} a_l d_l = \sum_{k=1}^{n}\sum_{l=1}^{n} c_k d_l (a_k b_l - a_l b_k) =$$
$$= \sum_{\substack{k,l\\k<l}} c_k d_l (a_k b_l - a_l b_k) - \sum_{\substack{k,l\\k<l}} c_l d_k (a_k b_l - a_l b_k) =$$
$$= \sum_{\substack{k,l\\k<l}} (c_k d_l - c_l d_k)(a_k b_l - a_l b_k) = |(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})|.$$

2. Let us consider n vectors $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n \,\epsilon\, X_n$. Components of vector $\bar{a}_i$ will be denoted by $a_{i1}, a_{i2}, \ldots, a_{in}$, $(i = 1, 2, \ldots, n)$. We put

$$\bar{a}_{1s} = \bar{a}_1 \times \bar{a}_{s+1}\,, \qquad \bar{a}_{k,s} = \bar{a}_{k-1,1} \times \bar{a}_{k-1,s+1}\,. \tag{3}$$

By simple induction we can prove

(a) The vector $\bar{a}_{k,s}$ is a function of $k+1$ vectors

$$\bar{a}_{k,s} = \bar{a}_{k,s}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k, \bar{a}_{k+s})\,. \tag{4}$$

Hence,

$$\bar{a}_{k,s}(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_k, \bar{a}_{k+t}) = \bar{a}_{k,t}\,.$$

(b):

(5) $$\bar{a}_{k,s}(\bar{a}_{l,1}, \dots, \bar{a}_{l,k}, \bar{a}_{l,k+s}) = \bar{a}_{k+l,s}(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{k+1}, \bar{a}_{k+1+s})\,.$$

3. Let us consider the product of the determinant

$$A(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n) = |a_{ik}|_n = \begin{vmatrix} a_{11}\, a_{12} \dots a_{1n} \\ a_{21}\, a_{22} \dots a_{2n} \\ \dots\dots\dots\dots \\ a_{n1}\, a_{n2} \dots a_{nn} \end{vmatrix}$$

and the analogous determinant B. From the well known Cauchy theorem we have

(6) $$AB = |(\bar{a}_i \cdot \bar{b}_k)|_n\,.$$

It is known [2], that if $a_{11} \neq 0$, then the determinant A of the degree $n \geqslant 2$ can be reduced to the determinant of degree $n-1$

(7) $$|a_{ik}|_n = \frac{1}{a_{11}^{n-2}} |a_{11}\, a_{ik} - a_{i1}\, a_{1k}|_{n-1}\,.$$

Hence, for the determinant AB, we obtain

$$AB = \frac{1}{(\bar{a}_1 \cdot \bar{b}_1)^{n-2}} |(\bar{a}_1 \cdot \bar{b}_1)(\bar{a}_i \cdot \bar{b}_k) - (\bar{a}_i \cdot \bar{b}_1)(\bar{a}_1 \cdot \bar{b}_k)|_{n-1}\,,$$

if $\bar{a}_1 \cdot \bar{b}_1 \neq 0$. But from (2) we have

$$AB = \frac{1}{(\bar{a}_1 \cdot \bar{b}_1)^{n-2}} |(\bar{a}_1 \times \bar{a}_i)(\bar{b}_1 \times \bar{b}_k)|_{n-1}\,.$$

Changing the indices and taking (3) into account we have

$$AB = \frac{1}{(\bar{a}_1 \bar{b}_1)^{n-2}} |\bar{a}_{1,i} \cdot \bar{\beta}_{1,k})|_{n-1}\,,$$

where functions $\beta_{1,k}$ are defined by (3).

If $(\bar{a}_{1,1}\bar{\beta}_{1,1}) \neq 0$, then after the next reduction we obtain

$$AB = \frac{1}{(\bar{a}_1 \bar{b}_1)^{n-2} (\bar{a}_{1,1} \bar{\beta}_{1,1})^{n-3}} |(\bar{a}_{2,i} \bar{\beta}_{2,k})|_{n-2}\,.$$

Hence, by induction, we obtain

(8) $$AB = \frac{1}{(\bar{a}_1 \bar{b}_1)^{n-2} (\bar{a}_{1,1}\, \bar{\beta}_{1,1})^{n-3} \dots (\bar{a}_{s-1,1} \bar{\beta}_{s-1,1})^{n-s-1}} |(\bar{a}_{s,i} \bar{\beta}_{s,k})|_{n-s}\,.$$

4. From (8) follow many interesting conclusions. If $s = n-2$, then

$$(9)\qquad AB = \frac{(\bar{a}_{n-1,1} \cdot \bar{\beta}_{n-1,1})}{(\bar{a}_1 \cdot \bar{b}_1)^{n-2}(\bar{a}_{11}\bar{\beta}_{11})^{n-3} \dots (\bar{a}_{n-3,1}\bar{\beta}_{n-3,1})}.$$

Hence, in the case $B = A$, we obtain

$$(10)\qquad |A| = \frac{a_{n-1,1}}{a_1^{n-2} a_{11}^{n-3} \dots a_{n-3,1}}.$$

Let us denote by

$$\begin{gathered} \bar{b}_1 = \bar{e}_1[1, 0, \dots, 0] \\ \bar{b}_2 = \bar{e}_2[0, 1, \dots, 0] \\ \dots\dots\dots\dots\dots\dots \\ \bar{b}_n = \bar{e}_n[0, 0, \dots, 1] \end{gathered}$$

unit-vectors of X_n. Then

$$\begin{gathered} \bar{\beta}_{11} = e_1'[1, 0, \dots, 0] \\ \bar{\beta}_{21} = e_1''[1, 0, \dots, 0] \\ \dots\dots\dots\dots\dots\dots \\ \bar{\beta}_{n-1,1} = \bar{e}_1^{(n-1)}[1, 0, \dots, 0]. \end{gathered}$$

Under these assumptions

$$(11)\qquad A = \frac{a^{\alpha'}_{n-1,1}}{a_{11}^{n-2} a_{11}^{\alpha' n-3} \dots a^{\alpha'}_{n-3,1}}$$

because $B = 1$ (by $a^{\alpha'}_{\nu,1}$ is denoted the first component of $\bar{a}_{\nu,1}$). It is easy to see that formula (11) gives us identities following in another way from Sylvester theorem [1].

5. Let us assume $B = A$. Then from (8) it follows that

$$(12)\qquad G_n(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n) = \frac{G_{n-s}(\bar{a}_{s,1}, \bar{a}_{s,2}, \dots, \bar{a}_{s,n-s})}{a_1^{2(n-2)} a_{11}^{2(n-3)} \dots a_{s-1,1}^{2(n-s-1)}},$$

where G_n is the Gram determinant

$$(13)\qquad A^2 = G_n(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n).$$

From Hadamard theorem [3] applied to determinant G_{n-s} we have

$$(14)\qquad |A| \leqslant \frac{a_{s,1} a_{s,2} \dots a_{s,n-s}}{a_1^{n-2}\; a_{11}^{n-3} \dots a_{s-1}^{n-s-1}}.$$

On the other hand, from (4) and (10), it follows that

$$(15)\qquad \frac{a_{s,\nu}}{a_1^{s-1} a_{11}^{s-2} \dots a_{s-2,1}} = [G_{s+1}(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_s, \bar{a}_{s+\nu})]^{1/2}.$$

We put

$$G_{s+1}(a_1, a_2, \ldots, a_s, a_{s+\nu}) = G_{s,\nu} \tag{16}$$

and in the particular case $\nu = 1$

$$G_{s,1} = G_{s+1}.$$

Now, from (5), (16) and (14), we obtain

$$|A| \leqslant \left(\frac{G_{s,1}, G_{s,2} \ldots G_{s,n-s}}{G_s^{n-s-1}}\right)^{1/2} \qquad s = 1, 2, \ldots, n-1. \tag{17}$$

Hadamard theorem can be obtained from (17) in the limit case $s = 0$. This means that the inequality (17) is an extension of this theorem.

Now, it will be proved, that if the number s increases, then the right hand of (17) decreases. It is sufficient to show that from (2) and (3) follows

$$a_{s+1,\nu} = |\bar{a}_{s,1} \times \bar{a}_{s,\nu+1}| \leqslant a_{s,1} a_{s,\nu+1}. \tag{18}$$

From (14) we can obtain in the analogous way, as above

$$|A| \leqslant \left(\frac{G_{s+1,1} G_{s+1,2} \ldots G_{s+1,n-s-1}}{G_{s+1}^{n-s-2}}\right)^{1/2} \leqslant \left(\frac{G_{s,1} G_{s,2} \ldots G_{s,n-s}}{G_s^{n-s-1}}\right)^{1/2}. \tag{19}$$

It means, that (17) is also a generalization of Hadamard theorem.

The obtained sequence of bounds is

$$|A| \leqslant \left(\frac{\prod_{\nu=1}^{2} G_{n-2,\nu}}{G_{n-2}}\right)^{1/2} \leqslant \left(\frac{\prod_{\nu=1}^{3} G_{n-3,\nu}}{G_{n-3}^2}\right)^{1/2} \leqslant \ldots \leqslant \left(\frac{\prod_{\nu=1}^{n-1} G_{1,\nu}}{a_1^{n-2}}\right)^{1/2} \leqslant \prod_{\nu=1}^{n} a_\nu.$$

6. The geometrical interpretation of these results is based on the notion of successive height of parallelepiped generated by vectors $\bar{a}_1, \bar{a}_2, \ldots \bar{a}_n$. This height h_ν is defined as a distance from the end point of $\bar{a}_\nu$ to the hyperplane generated by vectors $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{\nu-1}$.

It is known [4], that

$$h_\nu = \sqrt{\frac{G_\nu}{G_{\nu-1}}}. \tag{20}$$

From (10) and (11) we get

$$h_\nu = \frac{a_{\nu-1,1}}{a_1 a_{11} \ldots a_{\nu-2,1}}. \tag{21}$$

Comparing (21) with (10) we have

$$|A| = h_1 h_2 \ldots h_n. \tag{22}$$

By $H_{\nu-1,k}$ we shall denote the lateral height of order ν of the pa-

rallelepiped. The quantity $H_{\nu-1,k}$ is defined as a distance from the end point of the vector $\bar{a}_{k+\nu-1}$ to the hyperplane generated by vectors $\bar{a}_1$, $\bar{a}_2, \ldots, \bar{a}_{\nu-1}$ $(k = 2, 3, \ldots, n-\nu+1)$. Of course, $H_{\nu-1,1} = h_\nu$, if $k = 1$.

From (21) and (4) we have

$$H_{\nu-1,k} = \frac{a_{\nu-1,k}}{a_1 a_{11} \ldots a_{\nu-2,1}} \tag{23}$$

and from (18) and (23) we obtain

$$h_\nu = H_{\nu-1,1} \leqslant H_{\nu-2,2} \leqslant \ldots \leqslant H_{1,\nu-1} \leqslant a_\nu\,. \tag{24}$$

Thus, we have proved the following

THEOREM. *The lateral height of any order defined by the end point of $\bar{a}_\nu$ is not smaller than the successive height h_ν of the parallelepiped, and not greater than the lateral height of lower order.*

From (23) and (21) we can easily verify, that the inequality (14) and therefore (17) is equivalent to the following inequality:

$$|A| \leqslant h_1 h_2 \ldots h_{k+1} H_{k,2} H_{k,3} \ldots H_{k,n-k}\,. \tag{25}$$

7. The principal Fage inequality [5] is as follows

$$\begin{aligned} G_n(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)\, G_k(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k) &\leqslant \\ \leqslant G_{k+l}(\bar{a}_1, \bar{a}_2, \ldots \bar{a}_{k+l})\, G_{k+m}&(\bar{a}_1, \ldots, \bar{a}_k, \bar{a}_{k+l+1}, \ldots, \bar{a}_n) \end{aligned} \tag{26}$$

where $n = k+l+m$. It can easily be obtained from (22).

Indeed, if

$$G_n = H_1 H_2 \ldots H_n\,, \tag{27}$$

then

$$G_k = H_1 H_2 \ldots H_k \tag{28}$$

$$G_{k+l} = H_1 H_2 \ldots H_{k+l}\,. \tag{29}$$

From (22) we have

$$G_{k+m} = H_1 H_2 \ldots H_k H_{k,l+1}, H_{k+1,l+1}, \ldots, H_{k+m-1,l+1}\,. \tag{30}$$

But, on the other hand

$$\begin{gathered} H_{k+l+1} \leqslant H_{k,l+1} \\ H_{k+l+2} \leqslant H_{k+1,l+1} \\ \cdots\cdots\cdots\cdots \\ H_n \leqslant H_{k+m-1,l+1}\,. \end{gathered} \tag{31}$$

From (27)-(31) we obtain (26). This means, that from a geometrical point of view the Fage inequality can be obtained from (22) by changing successive heights of parallelepiped into lateral heights of smaller order.

8. The Wegner inequality [6]

$$A^2 \leqslant G_k(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)\, G_{n-k}(\bar{a}_{k+1}, \bar{a}_{k+2}, \ldots, \bar{a}_n) \tag{32}$$

can be obtained in the analogous way. Let us consider (28) and (27). From (22) we have

$$G_{n-k}(\bar{a}_{k+1}, \bar{a}_{k+2}, \ldots, \bar{a}_n) = H'_{1k} H'_{2k} \ldots H'_{n-k,k}, \tag{33}$$

where $H'_{1k}, H'_{2,k} \ldots H'_{n-k,k}$ are successive heights of parallelepiped generated by vectors $\bar{a}_{k+1}, \ldots, \bar{a}_n$ and simultaneously lateral heights of parallelepiped generated by vectors $\bar{a}_1, \ldots, \bar{a}_k$. But on the other hand

$$\begin{aligned} H_{k+1} &\leqslant H'_{1,k} \\ H_{k+2} &\leqslant H'_{2,k} \\ &\ldots\ldots\ldots\ldots\ldots \\ H_n &\leqslant H'_{n-k,k}. \end{aligned} \tag{34}$$

From (13), (33) and (34) follows immediately (32). This means that Wegner theorem is a particular case of Fage theorem.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 5. 1958

MATHÉMATIQUE

Sur l'extension de la notion de fonction rationnelle à l'espace euclidien n-dimensionnel

par

K. KURATOWSKI

Présenté le 27 février 1958

1. Introduction. Nous désignons, comme d'habitude, par $\mathcal{E}^n$ l'espace euclidien à n dimensions, par $\mathcal{S}_n$ la sphère formée des points $x_1^2+\ldots+x_{n+1}^2=1$ (nous l'identifions avec l'espace $\mathcal{E}^n$ augmenté du point à l'infini), par $\mathcal{P}_n$ l'espace $\mathcal{E}^n$ diminué du point 0. Par *unité* de l'espace $\mathcal{E}^n$ nous convenons d'entendre le point $1=(1,0,0,\ldots,0)$.

$\mathcal{Y}^{\mathcal{X}}$ désigne l'ensemble des transformations continues f de $\mathcal{X}$ en sous-ensembles de $\mathcal{Y}$. L'inclusion $f\subset g$ veut dire que la fonction g est une *extension* de f. Deux éléments f et g de $\mathcal{Y}^{\mathcal{X}}$ sont dits *homotopes* (rel. à $\mathcal{Y}$) lorsqu'il existe une fonction $h\in\mathcal{Y}^{\mathcal{X}\times\mathcal{J}}$ ($\mathcal{J}$ désignant l'intervalle fermé 01) telle que $h(x,0)=f(x)$ et $h(x,1)=g(x)$; nous écrivons dans ce cas $f\simeq g$. Pour les *fonctions partielles* $f|F$ et $g|F$, nous convenons d'écrire: $f\simeq g$ sur F, au lieu de $f|F\simeq g|F$.

L'ensemble $\mathcal{Y}^{\mathcal{X}}$ étant fixé, $[f]$ désigne la *classe d'homotopie* de la fonction f (c'est-à-dire l'ensemble de toutes les fonctions homotopes à f). En classifiant les fonctions-éléments de $\mathcal{Y}^{\mathcal{X}}$ en classes d'homotopie, on définit une décomposition de $\mathcal{Y}^{\mathcal{X}}$ en classes disjointes. Dans le cas où $\mathcal{Y}=\mathcal{P}_n$, nous désignons la famille de ces classes par $\mathfrak{B}_{n-1}(\mathcal{X})$.

La famille $\mathfrak{B}_1(\mathcal{X})$ a été l'objet d'une étude assez approfondie (cf. [3], [8] et [7], § 55). On montre, en particulier, que $\mathfrak{B}_1(\mathcal{X})$ est un groupe abélien par rapport à la multiplication des classes d'homotopie, définie par l'équivalence

(1) $\{[f_1]\cdot[f_2]=[f_3]\}\equiv\{f_1(x)\cdot f_2(x)=f_3(x)\quad \text{quel que soit } x\in\mathcal{X}\}$,

la multiplication dans le membre droit étant la multiplication habituelle des nombres complexes.

Dans le cas où $\mathcal{X}$ est un sous-ensemble compact F de $\mathcal{E}^2$, ce théorème peut être précisé comme soit:

Théorème I_2. *En désignant par $R_0, R_1, \ldots$ la suite (finie ou infinie) des composantes de $\mathcal{S}_2 - F$ et en posant $p_i \in R_i$ pour $i = 1, 2, \ldots$ (R_0 désignant la composante non bornée), les classes d'homotopie des translations $[(x-p_1)|F], [(x-p_2)|F], \ldots$ sont les générateurs du groupe $\mathfrak{B}_1(F)$.*

En conséquence: *suivant que le nombre des composantes de $\mathcal{S}_2 - F$ est fini, soit $m+1$, ou infini, on a les isomorphies:*

$$(2) \qquad \mathfrak{B}_1(F) \underset{\text{gr}}{=} \mathcal{G}^m \quad \textit{respectivement} \quad \mathfrak{B}_1(F) \underset{\text{gr}}{=} \mathcal{G}^\omega,$$

$\mathcal{G}$ désignant le groupe d'entiers et $\mathcal{G}^\omega$ — le groupe additif des suites infinies d'entiers ne contenant qu'un nombre fini de termes non nuls.

Nous allons établir dans cette note le théorème général I_n qui s'obtient de I_2 en remplaçant $\mathcal{S}_2$ par $\mathcal{S}_n$ et $\mathfrak{B}_1(F)$ par $\mathfrak{B}_{n-1}(F)$, F désignant un sous-ensemble compact arbitraire de $\mathcal{E}^n$ et la multiplication des classes d'homotopie pour $n > 2$ étant la multiplication cohomotopique définie par K. Borsuk (voir [1] et [2] *), cf. aussi la multiplication antérieurement définie par H. Freudenthal [4]).

Le cas où, à la place d'un ensemble compact F, on envisage un ensemble ouvert $G \subset \mathcal{E}^n$, sera considéré dans une note qui va paraître prochaînement **). Au lieu des classes d'homotopie on aura alors à considérer les *composantes* de l'espace $\mathcal{P}_n^G$ qui — pour F compact et aussi, pour G ouvert $\subset \mathcal{E}^2$ — coïncident avec les classes d'homotopie ***). Leur ensemble sera désigné par $\mathfrak{L}_{n-1}(G)$. Cet ensemble est un groupe abélien (par rapport à la multiplication cohomotopique) et — tout comme dans le cas du plan ([7], vol. II, p. 422, th. 5) — on a

$$(3) \qquad \mathfrak{L}_{n-1}(G) \underset{\text{gr}}{=} \mathcal{G}^m \quad \text{respectivement} \quad \mathfrak{L}_{n-1}(G) \underset{\text{gr}}{=} \mathcal{G}^{\aleph_0},$$

suivant que la famille des composantes de $\mathcal{S}_n - G$ est finie (formée de $m+1$ éléments) ou infinie; $\mathcal{G}^{\aleph_0}$ désigne le groupe additif de toutes les suites infinies à termes entiers.

D'autre part, $\mathfrak{L}_{n-1}(G)$ peut être considéré comme espace topologique, la notion de convergence de composantes étant induite par celle de convergence continue des fonctions-éléments de $\mathcal{P}_n^G$ (voir [7], vol. I, p. 93). L'isomorphie entre $\mathfrak{L}_{n-1}(G)$ et $\mathcal{G}^m$ (resp. $\mathcal{G}^{\aleph_0}$) est — comme je vais démontrer — en même temps une homéomorphie.

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*) La théorie de K. Borsuk n'étant pas directement applicable au cas où $F \subset \mathcal{E}^3$ et $\dim F = 3$ (puisqu'elle demande que $\dim F \leqslant 2n-4$), elle doit être complétée en ce point. Nous y retournerons à une autre occasion.

**) Pour d'autres applications des groupes de cohomotopie d'espaces compacts, voir [6].

***) $\mathcal{P}_n^G$ devient un espace topologique en entendant par convergence de ses éléments la convergence continue des fonctions.

C'est bien le cas de G ouvert $\subset \mathcal{E}^n$ qui est le but principal de nos recherches. Les théorèmes I_n-III_n (qui vont suivre) n'y jouent qu'un rôle auxiliaire. Bien que la différence entre ces théorèmes et les énoncés correspondants de Borsuk ([2], p. 227 et 240, cf. aussi [5], p. 146) ne soit pas essentielle (elle est plutôt de nature terminologique), il nous a paru désirable d'en donner ici des démonstrations directes en vue d'applications ultérieures et en vue de la méthode ("des fonctions rationnelles") dont nous allons nous servir.

Des nombreux théorèmes établis pour le plan (tels que les théorèmes de Runge, de Weierstrass, de Rouché [9], pp. 335, 350) pourront être étendus à l'espace à n dimensions à l'aide de cette méthode.

2. Multiplication cohomotopique. D'après la formule (1), la multiplication des classes d'homotopie, dans le cas où $F \subset \mathcal{E}^2$, est induite par celle des fonctions à valeurs complexes et celle-ci, à son tour, par celle des nombres complexes (éléments de $\mathcal{P}_2$). Dans le cas de $n>2$, il n'existe aucune multiplication de points de $\mathcal{P}_n$ (continue, commutative et qui satisfasse à (4)), sauf la multiplication restreinte au cas où l'un des facteurs est 1:

$$p\cdot 1 = p = 1\cdot p \quad \textit{où} \quad p \in \mathcal{P}_n. \tag{4}$$

Cette dernière multiplication donne lieu à la multiplication des fonctions $f_1, f_2 \in \mathcal{P}_n^F$ dans le cas particulier où elles satisfont à la condition suivante (et où elles seront dites *multipliables*):

$$\textit{quel que soit } x \in F, \textit{ on a soit } f_1(x) = 1, \textit{ soit } f_2(x) = 1. \tag{5}$$

Cela suffit pour définir la multiplication des classes d'homotopie dans le cas où *F est compact* $\subset \mathcal{E}^n$. Car, dans cette hypothèse on a les théorèmes importants suivants de K. Borsuk (voir [2], p. 235 et p. 237):

2.1. *A tout couple $f_1, f_2 \in \mathcal{P}_n^F$ correspond un couple $f_1', f_2' \in \mathcal{P}_n^F$, multipliable et tel que*

$$f_1' \simeq f_1 \quad \textit{et} \quad f_2' \simeq f_2. \tag{6}$$

2.2. *Si f_1' et f_2' sont multipliables, ainsi que f_1'' et f_2'', on a l'implication*

$$(f_1' \simeq f_1'' \quad \textit{et} \quad f_2' \simeq f_2'') \Rightarrow (f_1'\cdot f_2' \simeq f_1''\cdot f_2''). \tag{7}$$

On peut donc — afin de définir le produit $\Gamma_1\cdot\Gamma_2$ de deux classes d'homotopies — appliquer la formule (1) en représentant Γ_1 et Γ_2 sous la forme $\Gamma_1 \equiv [f_1]$ et $\Gamma_2 = [f_2]$ où les fonctions f_1 et f_2 sont multipliables.

La multiplication des classes d'homotopie étant ainsi conçue, on confère à l'ensemble $\mathfrak{B}_{n-1}(F)$ le caractère d'un groupe abélien ([1], [2], p. 239

et [9], p. 211) *). L'élément neutre de ce groupe est celui qui contient les fonctions constantes. On définit l'élément $1:\Gamma$ en convenant que:

$$si \quad p=(x_1,\ldots,x_n)\neq 0, \quad 1:p=\frac{1}{|p|}\cdot(x_1,\ldots,x_{n-1},-x_n).$$

Pour des raisons techniques, convenons d'écrire

$$(8) \qquad f_0 \simeq f_1\cdot f_2 \quad \text{dès que} \quad f_0\in\Gamma_1\cdot\Gamma_2, \quad f_1\in\Gamma_1, \quad f_2\in\Gamma_2,$$

donc aussi pour les fonctions f_1 et f_2 non multipliables.

Cela nous permettra de nous servir d'un calcul qui ne diffère du calcul habituel de multiplication que par l'emploi du signe $\simeq$ au lieu de $=$. On peut, par exemple, multiplier toujours les homotopies suivant la règle (7); on peut faire correspondre à tout couple f_1, f_2 une fonction f_3 telle que $f_3\simeq f_1\cdot f_2$, ainsi qu'une fonction $f_4\simeq f_1:f_2$, etc. L'homotopie d'une fonction $f\in\mathcal{P}_n^F$ à une fonction rationnelle, c'est-à-dire une relation de la forme

$$(9) \qquad f(x)\simeq(x-p_1)^{k_1}\cdot\ldots\cdot(x-p_m)^{k_m}$$

a donc aussi toujours un sens (pourvu que $p_i\in\mathcal{E}^n-F$).

Notons encore les propriétés importantes suivantes de la multiplication cohomotopique ([2], [9]):

2.3. *Si* $F_0=\overline{F}_0\subset F$, *on a*

$$(f_1\cdot f_2)|F_0\simeq(f_1|F_0)\cdot(f_2|F_0) \quad \text{et} \quad (f_1:f_2)|F_0\simeq(f_1|F_0):(f_2|F_0).$$

2.4. *h étant une transformation homéomorphe d'un espace F^* dans F, $f_0\simeq f_1\cdot f_2$ entraîne $f_0h\simeq f_1h\cdot f_2h$.*

2.5-6. *Si $F=\mathcal{S}_{n-1}$, à tout f correspond un entier k, tel que $f(x)\simeq x^k$; de plus, si $x^m\simeq 1$, on a $m=0$* **).

En vue d'applications, citons la proposition suivante qui est une conséquence directe de 2.4-6.

2.7-8. *F désignant la surface d'une boule (à n dimensions) de centre p, on a $f(x)\simeq(x-p)^k$; de plus, on n'a $(x-p)^m\simeq 1$ que dans le cas où $m=0$.*

3. Théorèmes auxiliaires sur l'homotopie dans $\mathcal{E}^n$. Soit F un sous-ensemble compact de $\mathcal{E}^n$. Soit $f\in\mathcal{P}_n^F$. On a les théorèmes suivants:

3.1. *Si $f\simeq 1$, il existe une fonction $f^*\in\mathcal{P}_n^{\mathcal{E}^n}$ telle que $f\subset f^*$, donc que $f^*\simeq 1$.*

---

*) Le groupe $\mathfrak{B}_{n-1}(F)$ devient identique au $n-1$-ème groupe de cohomotopie de Borsuk (désigné par $\pi^{n-1}(F)$) en remplaçant $\mathcal{P}_n$ par $\mathcal{S}_{n-1}$ (ce qui, au point de vue topologique, ne présente pas de différence essentielle, puisque $\mathcal{P}_n$ est le produit de $\mathcal{E}$ et de $\mathcal{S}_{n-1}$).

**) k est bien l'ordre de la transformation f au sens de Brouwer.

Cf. [7], vol. II, p. 280, th. 7. L'homotopie $f^* \simeq 1$ résulte du fait que, pour toute fonction $g \in \mathcal{P}_n^{\mathcal{E}^n}$, on a $g \simeq 1$.

3.2. *Si $\mathcal{S}_n - F$ est connexe, on a $f \simeq 1$* (cf. [7], v. II, p. 347, th. 10).

En particulier, *si F est une boule (à n dimensions) et $p \in \mathcal{E}^n - F$, on a $x - p \simeq 1$ sur F.*

3.3. *Si R est une composante de $\mathcal{E}^n - F$ telle que $f \simeq 1$ sur* $\mathrm{Fr}(R)$, *il existe une fonction $f^* \in \mathcal{P}_n^{F \smile R}$ telle que $f \subset f^*$* (cf. (7), v. II, p. 346, th. 9).

3.4. *Toute fonction f admet une extension $f^* \in \mathcal{P}_n^{\mathcal{S}_n - Z}$ où Z est un ensemble fini convenablement choisi* (cf. ibid.).

3.5. *Si les points p et q appartiennent à la même composante de $\mathcal{E}^n - F$, on a $x - p \simeq x - q$ sur F.*

En effet, pq étant un arc contenu dans $\mathcal{E}^n - F$ et g une transformation continue de l'intervalle $0 \leqslant t \leqslant 1$, sur pq, on pose $h(x, t) = x - g(t)$.

3.6. *Soit F_1 un sous-ensemble fermé de F qui sépare tout couple de points de $\mathcal{S}_n - F$ séparé par F. Si $f \simeq 1$ sur F_1, on a $f \simeq 1$ (sur F).*

En particulier: *si $f \simeq 1$ sur* $\mathrm{Fr}(F)$, *on a $f \simeq 1$ sur F.*

Soit, en effet, R une composante de $\mathcal{E}^n - F$. En tenant compte de 3.3 et 3.2, tout revient à montrer que

$$f \simeq 1 \quad \text{sur} \quad \mathrm{Fr}(R)\,. \tag{10}$$

Comme $\mathcal{E}^n - F \subset \mathcal{E}^n - F_1$, il existe une composante T de $\mathcal{E}^n - F_1$ qui contient R. Il vient $R = T - F$, car en supposant que $p \in T - F - R$ et que $q \in R$, les points p et q seraient séparés par F mais non par F_1, contrairement à l'hypothèse.

En remplaçant dans 3.1 f par $f|F_1$, il vient

$$(f|F_1) \subset g \in \mathcal{P}_n^{\mathcal{E}^n}\,. \tag{11}$$

Posons: $h(x) = g(x)$ pour $x \in \mathcal{E}^n - T$ et $h(x) = f(x)$ pour $x \in F \frown \overline{T}$.

Comme $(\mathcal{E}^n - T) \frown (F \frown \overline{T}) \subset \overline{T} - T \subset F_1$, on conclut de (11) que h est une fonction continue définie sur l'ensemble

$$(\mathcal{E}^n - T) \smile (F \frown \overline{T}) = (\mathcal{E}^n - T) \smile F = \mathcal{E}^n - R\,, \tag{12}$$

puisque $R = T - F$. Autrement dit, $h \in \mathcal{P}_n^{\mathcal{E}^n - R}$, donc $h \simeq 1$ (selon 3.2). L'homotopie (10) en résulte, car les formules $\mathrm{Fr}(R) \subset F$ et $\mathrm{Fr}(R) \subset \overline{R} \subset \overline{T}$ impliquent en vertu de la définition de h que $f(x) = h(x)$ pour $x \in \mathrm{Fr}(R)$.

4. Démonstration du théorème I_n. La démonstration du théorème I_n se ramène à celle des théorèmes II_n et III_n suivants:

THÉORÈME II_n. *Etant donné un sous-ensemble compact F de $\mathcal{E}^n$, toute fonction $f \in \mathcal{P}_n^F$ est homotope à une fonction rationnelle (dans le sens de la relation (9)).*

Plus précisement, *R_i et p_i ayant le même sens que dans le th.* I_n, *il existe un système d'entiers:* $m, k_1, \ldots, k_m$, *tel que la relation* (9) *a lieu; ou bien — ce qui est équivalent — que l'égalité suivante a lieu*

$$[f]=[(x-p_1)|F]^{k_1}\cdot\ldots\cdot[(x-p_m)|F]^{k_m}. \tag{13}$$

En outre, *si* $\mathcal{S}_n-F=R_0$, *on a* $f\simeq 1$ *(donc* $m=0$ *et* $[f]=[1]$).

Théorème III_n. *Les exposants $k_1, \ldots, k_m$ dans le th.* II_n *sont déterminés de façon univoque: ils ne dépendent, pour f fixe, que des composantes respectives de* $\mathcal{S}_n-F$.

Démonstration du théorème II_n. D'après 3.4 et 2.3, la démonstration se ramène au cas où F est un continu "élémentaire":

$$C_m=\mathcal{S}_n-(Q_0\cup\ldots\cup Q_m) \quad \text{où} \quad \bar{Q}_i\cap\bar{Q}_{i'}=0 \quad \text{pour} \quad i\neq i' \tag{14}$$

et où $Q_0, \ldots, Q_m$ sont des boules ouvertes à n dimensions, dont Q_0 est non-bornée.

Soit $p_i\in Q_i$. Posons $F=C_m$, $m\geqslant 0$, et procédons par induction relativement à m.

1) Si $m=0$, on a $C_0=\mathcal{S}_n-Q_0$, c'est-à-dire $\mathcal{S}_n-C_0=Q_0$ et par conséquent $f\simeq 1$ selon 3.2.

2) Soit $m>0$ et admettons que le théorème est vrai pour $m-1$. Posons, pour abréger, $A_m=\bar{Q}_m-Q_m$. D'après 2.7, il existe un entier k_m, tel que

$$f(x)\simeq(x-p_m)^{k_m} \quad \text{sur} \quad A_m. \tag{15}$$

Comme $x-p_m\neq 0$ pour $x\in C_m$, on a $(x-p_m|C_m)\in\mathcal{P}_n^{C_m}$ et il existe, par conséquent, une fonction $g\in\mathcal{P}_n^{C_m}$, telle que

$$g(x)\simeq f(x)\cdot(x-p_m)^{-k_m} \quad \text{sur} \quad C_m. \tag{16}$$

D'après (15), $g\simeq 1$ sur A_m. Il existe donc selon 3.3 une fonction $g^*\in\mathcal{P}_n^{C_m\cup Q_m}$ telle que $g\subset g^*$. Comme $C_m\cup Q_m=C_{m-1}$, il vient $g^*\in\mathcal{P}_n^{C_{m-1}}$. Le théorème étant supposé vrai pour $m-1$, on a

$$g^*(x)\simeq(x-p_1)^{k_1}\cdot\ldots\cdot(x-p_{m-1})^{k_{m-1}} \quad \text{sur} \quad C_{m-1}. \tag{17}$$

Comme $g^*(x)=g(x)$ pour $x\in C_m$, il résulte de (17) (en vertu de 2.3) que

$$g(x)\simeq(x-p_1)^{k_1}\cdot\ldots\cdot(x-p_{m-1})^{k_{m-1}} \quad \text{sur} \quad C_m. \tag{18}$$

En divisant (16) par (18), on obtient l'homotopie (9) (sur C_m).

Démonstration du théorème III_n. Nous allons établir d'abord le lemme suivant:

Lemme. *C_m étant le continu élémentaire défini par la formule* (14) *et p_i étant un point de Q_i, les classes d'homotopie* $[(x-p_1)|C_m], \ldots, [(x-p_m)|C_m]$

sont linéairement indépendantes; autrement dit, on a l'implication:

$$\{(x-p_1)^{k_1}\cdot\ldots\cdot(x-p_m)^{k_m}\simeq 1 \quad \textit{sur} \quad C_m\}\Rightarrow\{k_1=0,\ldots,k_m=0\}.$$

Il est évidemment légitime d'admettre que $m\geqslant 1$. En posant $A_m=\bar{Q}_m-Q_m$, on a par hypothèse

(19) $$(x-p_1)^{k_1}\cdot\ldots\cdot(x-p_m)^{k_m}\simeq 1 \quad \text{sur} \quad A_m.$$

D'autre part, d'après 3.2, on a

(20) $$x-p_1\simeq 1,\ldots,x-p_{m-1}\simeq 1 \quad \text{sur} \quad \bar{Q}_m, \quad \text{donc sur} \quad A_m.$$

Les formules (19) et (20) donnent $(x-p_m)^{k_m}\simeq 1$ sur A_m, d'où $k_m=0$ selon 2.8. On a donc de façon générale: $k_i=0$ pour $i=1,\ldots,m$.

Le lemme étant ainsi établi, passons au th. III_n.

Il s'agit de démontrer que les conditions:

(21) $$f(x)\simeq(x-p_1)^{k_1}\cdot\ldots\cdot(x-p_m)^{k_m} \quad \text{et} \quad f(x)\simeq(x-p_1)^{j_1}\cdot\ldots\cdot(x-p_m)^{j_m}$$

(sur F) impliquent $k_1=j_1,\ldots,k_m=j_m$.

(Nous admettons que les zéros et pôles $p_1,\ldots,p_m$ sont les mêmes pour les deux fonctions rationnelles, ceci étant légitime en vertu de 3.5; il peut arriver d'ailleurs que parmi les exposants il y en ait qui s'annulent).

En admettant que C_m est défini par les conditions (14) et que $Q_i\subset R_i$, on a $F\subset C_m$, et F sépare tout couple de points de $\mathcal{S}_m-C_m$ qui est séparé par C_m. On en conclut en raison de 3.6 que l'homotopie

$$(x-p_1)^{k_1-j_1}\cdot\ldots\cdot(x-p_m)^{k_m-j_m}\simeq 1,$$

qui a lieu sur F (d'après (21)), a lieu aussi sur C_m.

D'après le lemme, $k_i=j_i$ pour $i=1,\ldots,m$.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 5. 1958

MATHEMATICS

A Localization Theorem for Multiplicative Linear Functionals

by

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Presented by W. ORLICZ on February 28, 1958

Let T be a completely regular Hausdorff space and let $\boldsymbol{R}$ be a σ-ideal of boundary subsets of T (i. e. a σ-additive and hereditary family of subsets of T such that no open non-void set belongs to $\boldsymbol{R}$). Let $\boldsymbol{m}(T)$ be the family of all real-valued bounded functions $x = x(t)$, defined for $t \in T$, and let Y be a linear subset of $\boldsymbol{m}(T)$ containing constant functions, closed with respect to the operation supremum $x_1 \vee \ldots \vee x_n$ of a finite number of functions and closed with respect to uniform convergence.

Typical examples satisfying the above conditions are as follows:

A. Y is the class of all real-valued bounded continuous functions on T, and $\boldsymbol{R}$ consists of the empty set only.

B. $T = \langle 0, 1\rangle$, Y is the class of all bounded measurable functions, and $\boldsymbol{R}$ is the ideal of the sets of measure zero.

C. T is a complete metric space, Y is the class of all real bounded functions satisfying the condition of Baire ([2], p. 306) and $\boldsymbol{R}$ is the ideal of the sets of the first category.

D. $T = \langle 0, 1\rangle$, Y is the class of all Riemann-integrable functions, and R is the ideal of the sets of measure zero.

Let us denote by *the* $\boldsymbol{R}$*-essential supremum* the number

$$\sup_{A}{}_{R}\, x(t) = \inf \big\{a : A \cap \{t : x(t) > a\} \in \boldsymbol{R}\big\} = \inf_{B \in R} \sup \{x(t) : t \in A \setminus B\} .$$

(if $A \in \boldsymbol{R}$, then we assume $\sup_{A}{}_{R}\, x(t) = 0$ for every x); next, let us denote by $\overline{\lim_{t \to t_0}}{}_{R}\, x(t)$ the number

$$\inf \{\sup_{U \setminus t_0}{}_{R}\, x(t) : t_0 \in \operatorname{Int}(U)\}$$

(if t_0 is isolated in T, we assume $\overline{\lim_{t \to t_0}}{}_{R}\, x(t) = x(t)$), and let $\underline{\lim_{t \to t_0}}{}_{R}\, x(t) = -\overline{\lim_{t \to t_0}}{}_{R} \big(-x(t)\big)$.

We shall write $x(t) \geqslant_R y(t)$ on A ($x(t) \geqslant y(t)$ *for* **R**-*almost* all $t \in A$), if there exists a set $B \in \boldsymbol{R}$ such that $x(t) \geqslant y(t)$ for $t \in A \setminus B$. Similarly we define the expressions $x(t) >_R y(t)$ and $x(t) =_R y(t)$. Finally, let us define $T_R = \bigcup_{A \in \boldsymbol{R}} A$.

Assuming that the functions $x(t)$ from Y are defined on T up to a set from $\boldsymbol{R}$, we obtain the quotient space Y/R.

THEOREM 1. *Y/R is an M-space with a unit in the sense of S. Kakutani* [1], *with the norm*

$$\|x\| = \sup_T{}_R |x(t)|$$

and the ordering

$x \geqslant y$, if there exist functions $x(t)$ and $y(t)$ from Y corresponding to the equivalence classes x, y, such that $x(t) \geqslant_R y(t)$ on T.

Following Kakutani, let us denote by $\boldsymbol{\Omega}$ the set of all positive linear functionals over Y/R, of norm 1, satisfying the implication

$$x \wedge y = 0 \Rightarrow \xi(x) \cdot \xi(y) = 0 .$$

$\boldsymbol{\Omega}$ is a compact Hausdorff space in the weak topology of functionals induced by $(Y/R)^*$, and Y/R is equivalent (in the linear, metric and lattice sense) to the space $C(\boldsymbol{\Omega})$ of real continuous functions on $\boldsymbol{\Omega}$ (with the usual norm and ordering).

On the other hand, Y/R is at the same time a ring and, x, y being the equivalence classes of functions $x(t)$, $y(t)$, respectively, the class $x \cdot y$ corresponds to the function $x(t) \cdot y(t)$. It is known that $\boldsymbol{\Omega}$ may also be identified with the set of all multiplicative linear functionals ($\neq 0$) over Y/R.

Definition 1. A point $t_0 \in T$ will be termed the localization point of a linear functional ξ (over Y/R) if for every $x, y \in Y/R$ and for every neighbourhood U of t_0 the condition $x(t) =_R y(t)$ on U implies $\xi(x) = \xi(y)$.

$L(t_0)$ will denote the set of all functionals of $\boldsymbol{\Omega}$ for which t_0 is a localization point. Obviously, $L(t_0)$ is closed in $\boldsymbol{\Omega}$.

THEOREM 2. *Every point t of T is a localization point of a multiplicative linear functional ξ (i. e. $L(t) \neq 0$ for all $t \in T$).*

THEOREM 3 (Localization theorem). *If T is compact, then every multiplicative linear functional over Y/R has at least one localization point in T.*

THEOREM 4. *Given t_0, every functional $\xi \in L(t_0)$, except eventually the functional $\xi_0(x) = x(t_0)$ (ξ_0 being defined only if $t_0 \in T \setminus T_R$), satisfies the condition*

$$(1) \qquad \underline{\lim}_{t \to t_0}{}_R\, x(t) \leqslant \xi(x) \leqslant \overline{\lim}_{t \to t_0}{}_R\, x(t) \quad \textit{for every} \quad x \in Y/R\,^{*)};$$

conversely, $\xi \in \boldsymbol{\Omega}$ and (1) *imply $\xi \in L(t_0)$.*

---

*) Thus, Theorem 2 is a generalization of Theorem 1 of [3].

THEOREM 5. *For any* $t_0 \in T$ *and* $x, y \in Y/R$ *there exist functionals* $\xi, \eta \in L(t_0)$ *such that*

$$\overline{\lim_{t\to t_0}}{}_R\, x(t) = \xi(x) \quad \text{and} \quad \underline{\lim_{t\to t_0}}{}_R\, x(t) = \eta(x)\,.$$

Definition 2. We shall say that Y/R separates the points u and v of T if there exists an x in Y/R such that

$$\overline{\lim_{t\to u}}{}_R\, x(t) < \underline{\lim_{t\to v}}{}_R\, x(t)\,.$$

THEOREM 6. *Y/R separates u and v if, and only if,* $L(u) \cap L(v) = 0$.

Definition 3. We shall say that Y/R separates weakly u and v if there exists an x in Y/R such that

$$\underline{\lim_{t\to u}}{}_R\, x(t) \neq \overline{\lim_{t\to v}}{}_R\, x(t)$$

or

$$x(u) \neq x(v) \qquad (\text{if } u, v \in T \setminus T_R)\,,$$

$$\overline{\lim_{t\to u}}{}_R\, x(t) < x(v) \qquad (\text{if } u \in T_R,\ v \in T\setminus T_R)\,,$$

$$x(u) > \overline{\lim_{t\to v}}{}_R\, x(t) \qquad (\text{if } u \in T\setminus T_R,\ v \in T_R)\,.$$

THEOREM 7. *Y/R separates weakly u and v if, and only if,* $L(u) \neq L(v)$.

THEOREM 8. *Let T be compact and let Y/R separate all the pairs of different points of T. Then, for every real-valued continuous function $x(t)$ on T, there exists an y in Y such that $x(t) =_R y(t)$ on T.*

In the above sense we shall say that Y/R contains the space $C(T)$ of continuous functions on T *).

THEOREM 9. *Under the hypotheses of Theorem 8, the mapping*

$$\varphi(\xi) = t \qquad (t \in T,\ \xi \in L(t)),$$

is a continuous mapping of Ω onto T, corresponding to Stone's natural mapping of the set Ω of multiplicative linear functionals $(\neq 0)$ over Y/R onto the set Ω_0 of all multiplicative linear functionals $(\neq 0)$ over the subring $C(T)$ ([4], p. 475). *Moreover, $L(t) = \varphi^{-1}(t)$ for all $t \in T$.*

Thus the decomposition $\Omega = \bigcup_{t\in T} L(t)$ is semicontinuous.

THEOREM 10. *Under the hypotheses of Theorem 8, every linear functional ξ over Y/R with t_0 as its localization point is of the form*

$$\xi(x) = \int_{L(t_0)} \mathbf{x}(\eta)\, d\mu\,,$$

*) Evidently, $\sup_T x(t) = \sup_{T}{}_R\, x(t)$ for all continuous function on T.

where $\boldsymbol{x}$ *is the element of* $C(\boldsymbol{\Omega})$ *corresponding to* x *in the representation of Kakutani, and* μ *is a signed Borel measure on* $\boldsymbol{\Omega}$.

THEOREM 11. *Under the hypotheses of Theorem* 8, *the set of all linear combinations of the functionals of* $L(t_0)$ *is dense in the *-weak topology in the set of all linear functionals localized at* t_0.

The proofs and a detailed analysis of the problem will be published in Studia Mathematica.

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BULLETIN DE L'ACADÉMIE
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et phys. — Vol. VI, No. 5, 1958

MATHEMATICS

An Extension to Locally Convex Spaces of Borsuk's Theorem on Antipodes

by

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Presented by K. BORSUK on March 7, 1958

The known theorem on antipodes of K. Borsuk [1] (cf. [2]) was extended by M. A. Krasnoselsky [3] to completely continuous operators in Banach spaces.

In the present note we shall prove an extension of this theorem to the case of completely continuous transformations in locally convex linear topological spaces.

Let X be a locally convex linear topological space, i. e. a linear Hausdorff space with the topology defined by means of a system of neighbourhoods of the origin 0, which are symmetric convex open sets [5], [6]. Let U be an open convex symmetric neighbourhood of the origin. The pseudonorm $|x|_U$ corresponding to U is defined by $|x|_U = \inf\{t | t > 0, x \in tU\}$. It follows easily that $|x|_U$ is a non-negative function with the following properties: $|tx|_U = |t|\,|x|_U$ for any real t;

$$|x+y|_U \leqslant |x|_U + |y|_U \quad \text{for any } x, y \text{ of } X \text{ and } |x|_U < 1,$$

if, and only if, $x \in U$.

A transformation F from a subset M of X into X is called completely continuous on M, if F is continuous on M, and the image $F(M)$ of M is a subset of a bicompact set in X.

Let G be an open set in X and F a completely continuous transformation of G into X. Let a be a point not on $f(\bar{G}-G)$, where $\bar{G}$ is the closure of G and $f(x) = x - F(x)$. It is known that the Leray-Schauder degree of mapping f in X has a defined meaning $d[f, G, a]$. (For a detailed discussion of the degree in locally convex linear topological spaces see [4]).

THEOREM 1. *Let U be an open convex symmetric neighbourhood of the origin. Let us assume that a continuous mapping $f(x) = x + F(x)$ is defined on $\overline{U}$, where F is a completely continuous transformation from $\overline{U}$*

into X. If $f(x)$ does not vanish on the boundary of U, i. e. 0 non $\epsilon\, f(\bar{U}-U)$, and if condition

(1) $$f(x) = -f(-x) \quad \textit{for every } x \textit{ of } \bar{U}-U$$

is satisfied, then degree $d[f, U, 0]$ is odd.

Proof. Since the origin is not contained in $f(\bar{U}-U)$, there exists an open convex symmetric neighbourhood V of the origin such that the $V \cap f(\bar{U}-U)$ is empty. Let V_1 be an open convex symmetric neighbourhood of the origin such that the direct sum $V_1 \oplus V_1$ is contained in V. By hypothesis, $F(\bar{U})$ is contained in a bicompact set K; hence, there exists a finite number of points p_i of K such that

(2) $$\bigcup_{i=1}^{m} V_1(p_i) \supset K,$$

where $V_1(p_i) = p_i \oplus V_1$ for $i = 1, 2, \ldots, m$.

In the neighbourhood $V_1(p_i)$ there exists a point q_i such that

(3) $$|q_i|_U \neq 0 \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Since $q_i \in p_i \oplus V_1$, and $V_1 \oplus V_2 \subset V$, we have

(4) $$\bigcup_{i=1}^{m} V(q_i) \supset K.$$

Let E^n be the linear manifold spanned by elements $q_1, q_2, \ldots, q_m$. We put $u_i(x) = \max\{(1-|x-q_i|_V), 0\}$ and define a continuous transformation S_n by

(5) $$S_n(x) = \Big(\sum_{i=1}^{2m} \mu_i(x)\Big)^{-1} \sum_{i=1}^{2m} \mu_i(x) q_i,$$

where $q_{m+k} = -q_k$ for $k = 1, 2, \ldots, m$.

It follows from (4) that

$$\bigcup_{i=1}^{2m} V(q_i) \supset K.$$

Hence,

(6) $$S_n(x) - x \in V \quad \text{for any} \quad x \text{ of } K.$$

Putting $f_n(x) = x - S_n f(x)$ for $x \in \bar{U}$ we obtain, by (5),

(7) $$f(x) - f_n(x) \in V \quad \text{for any} \quad x \text{ of } \bar{U}.$$

The space E^n being of finite dimension is Euclidean, and f_n transforms $\bar{U}^n = \bar{U} \cap E^n$ into E^n. It is easy to see, by (3), that pseudonorm $|x|_U$ is a norm on E^n, and the boundary $\bar{U}^n - U^n$ is the unit sphere $S^{(n-1)}$ of E^n. Since $f(x)$ does not vanish on $S^{(n-1)}$ neither does f_n, by (6). Thus,

we have $d[f, V, 0] = d[f_n, U^n, 0]$. It results from (5) that the transformation S_n is odd, i. e. $S(x) = -S_n(-x)$. Hence, we have, by (1),

$$f_n(x) = -f_n(-x) \quad \text{for any} \quad x \text{ of } S^{(n-1)}.$$

By Borsuk's theorem on antipodes we infer that the degree $d[f_n, U^n, 0]$ is odd, and thus Theorem 1 is proved.

COROLLARY 1. *If the conditions of Theorem* 1 *are satisfied, F possesses a fixed point in U, i. e. there exists an element x_0 of U such that $F(x_0) = x_0$.*

THEOREM 2. *Let us assume the conditions of Theorem* 1 *are satisfied provided that condition* (1) *is replaced by condition*

(8) $\quad f(x) \neq tf(-x)$ *for any* t $(0 \leqslant t \leqslant 1)$ *and* x *of* $\overline{U}-U$.

Then the assertion of Theorem 1 *holds.*

Proof. Consider the transformation

$$f_t(x) = \frac{1}{1+t} f(x) - \frac{t}{1+t} f(-x) \quad \text{for} \quad 0 \leqslant t \leqslant 1 .$$

It follows from (8) that $f_t(x)$ does not vanish on $\overline{U}-U$. We have $f_0(x) = f(x)$ and f_1 satisfies condition (1). By Theorem 1, the degree $d[f_1, U, 0]$ is odd. Since $x - f_t(x)$ is continuous in (x, t) and lies for all t in the fixed bicompact subset of X, $d[f_t, U, 0]$ is constant on $[0, 1]$.

COROLLARY 2. *If the conditions of Theorem* 2 *are satisfied, then F possesses a fixed point in U, i. e. there exists an element x_0 of U such that $F(x_0) = x_0$.*

As a particular case of Theorem 2 we obtain Krasnoselsky's extension to Banach spaces of Borsuk's theorem on antipodes mentioned above.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 5. 1958

MATHEMATICS

Continuous Transformations of Open Sets in Locally Convex Spaces

by

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Presented by K. BORSUK on March 7, 1958

In paper [2] K. Borsuk proved some important properties of a continuous transformation $f(x)$ of a finite-dimensional Euclidean space X into itself provided that $f(x)$ is an ε-mapping or an ε-mapping in the narrow sense [1]. Transformation $f(x)$ is said to be an ε-mapping if

$$f(x') = f(x'') \quad \text{for} \quad x', x'' \text{ of } X$$

implies

$$\|x' - x''\| < \varepsilon .$$

In paper [4] A. Granas gave an extension to an arbitrary Banach space X of some results of Borsuk [2], under the additional condition of transformation $f(x)$ being of the form $f(x) = x - F(x)$, where $F(x)$ is a completely continuous transformation of X into itself.

Paper [1] contains a further examination of this transformation in Banach spaces. For this purpose the notion of a local ε-mapping in the narrow sense is introduced in [1].

Using the notion of a local ε-mapping, A. Granas [5] proved a general theorem on completely continuous displacement of an open set in an arbitrary Banach space which contains, in particular, Borsuk's theorem on ε-mapping, the Brower-Schauder theorem on invariance of domain, and Theorem 1 of paper [1].

In the present note we give a generalization of Borsuk's theorem on ε-mapping to a locally convex linear topological space. As a particular case of this theorem we obtain also a theorem on invariance of domain for completely continuous displacements in locally convex linear topological spaces. The results obtained in this note contain, in particular, the corresponding theorems mentioned above for Euclidean and Banach spaces.

For this purpose we introduce the notion of a ∂-mapping being a generalization of the notion of an ε-mapping. The main role in the proof is played by a generalization of Borsuk's theorem on antipodes for completely continuous displacements in locally convex linear topological spaces. This generalization is proved in the preceding paper on p. 297.

1. Let X be a locally convex linear topological space, i. e. a linear Hausdorff space with the topology defined by means of a system of neighbourhoods of the origin 0, which are symmetric convex open sets [5], [6].

Let $V(x) = x \oplus V$ be a neighbourhood of x, where V is an open convex symmetric neighbourhood of the origin and $x \oplus V$ is the direct sum of x and V, i. e. the set of all elements $x+y$ for fixed x and arbitrary y of V. Denote by $\bar{V}$ the closure of V. The symbol ∂V will stand for the boundary of V, i. e. $\partial V = \bar{V} - V$. A transformation f defined on $\bar{V}(x)$ with values in X is called a ∂-mapping on $\bar{V}(x)$, if for any z of $\bar{V}(x)$ the image

$$f(\bar{V}(x) \frown (z \oplus \partial V)) \quad \text{does not contain} \quad f(z).$$

This definition is equivalent to the following one: f is a ∂-mapping on V, if conditions

$$x', x'' \in \bar{V}(x) \quad \text{and} \quad f(x') = f(x'') \quad \text{imply } x' - x'' \text{ non} \in \partial V.$$

In this note we shall consider only completely continuous displacements on $\bar{G}$, i. e. continuous transformations f defined on $\bar{G}$ with values in X being of the form $f(x) = x - F(x)$, $x \in \bar{G}$, where F is a completely continuous transformation on $\bar{G}$. A transformation F from a subset M of X into X is said to be completely continuous if F is continuous on M, and the image $F(M)$ is a subset of a bicompact set in X.

If a is a point not on $f(\bar{G} - G)$, then the Leray-Schauder degree of mapping f in X has a defined meaning $d[f, G, a]$. (For a detailed discussion of the degree in locally convex linear topological spaces see [7]).

LEMMA. *Let U be an open convex symmetric neighbourhood of the origin and let f be a ∂-mapping on $\bar{U}$ of the form $f(x) = x - F(x)$, $x \in \bar{U}$, where F is a completely continuous transformation from $\bar{U}$ into X. Let us assume that neighbourhood U does not contain the whole straight line $\{t \cdot f(0)\}$ if $f(0) \neq 0$.*

Then the degree $d[f, G, f(0)]$ is odd.

Proof. Since f is a ∂-mapping on $\bar{U}$, we have $f(0)$ non $\in f(\partial U)$. For every x of $\bar{U}$ put

$$x'_t = \frac{x}{1+t} \quad \text{and} \quad x''_t = -\frac{tx}{1+t} \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

Since for any real $t(0 \leqslant t \leqslant 1)$ the points x'_t and x''_t belong to $\bar{U}$, one may define on $\bar{U}$ a family of completely continuous displacements $g_t(x)$ by the formula

$$g_t(x) = f(x'_t) - f(x''_t) = x - G_t(x)\,, \tag{1}$$

where

$$G_t(x) = F\left(\frac{x}{1+t}\right) - F\left(\frac{-tx}{1+t}\right).$$

Since f is a ∂-mapping on $\bar{U}$, we have

$$g_t(x) \neq 0 \quad \text{for every} \quad x \quad \text{of} \quad \partial U\,.$$

Since $G_t(x)$ is continuous in (x, t) and lies for all $t(0 \leqslant t \leqslant 1)$ in the fixed bicompact subset of X, the degree $d[g_t, U, 0]$ is constant on $[0, 1]$.

It follows from (1) that $g_1(x)$ satisfies the condition

$$-g_1(x) = g_1(-x) \quad \text{for any} \quad x \quad \text{of} \quad \partial U\,.$$

By Theorem 1 of [2], the degree $d[g_1, U, 0]$ is odd. But

$$g_0(x) = f(x) - f(0) \quad \text{and} \quad f(0)\ \text{non} \in f(\partial U)\,;$$

hence

$$d[g_0, U, 0] = d[f, U, f(0)]$$

and the lemma is proved.

Let G be an open set in X, and F a completely continuous transformation of G into X. Transformation f of the form $f(x) = x - F(x)$ is called a local ∂-mapping on G, if for every x of G there exists an open convex symmetric neighbourhood U_x of the origin such that:

1° the closure of $x \oplus U_x$ is contained in G;

2° f is a ∂-mapping on the closure of $x \oplus U_x$;

3° U_x does not contain the whole straight line

$$\{t \cdot f(x)\} \quad \text{if} \quad f(x) \neq 0\,.$$

THEOREM 1. *Let G be an open set in the locally convex linear topological space X, and f a continuous transformation of the form $f(x) = x - F(x)$, where F is a completely continuous transformation on G with values in X. If f is a local ∂-mapping on G, then the image $f(G)$ is an open set in X.*

Proof. Let $q = f(p)$ be an arbitrary point of $f(G)$. We shall show that there exists a neighbourhood of q which is contained in $f(G)$. By hypothesis there exists an open convex symmetric neighbourhood U_p of the origin satisfying conditions 1°-3° for $x = p$. Put $y = x - p$ and define a continuous transformation $\bar{f}(y) = y - \bar{F}(y)$ for $y \in \bar{U}_p$, where $\bar{F}(y) = F(x) - p$. We have

$$\bar{f}(y) = f(x) \tag{2}$$

and $\bar{F}(y)$ is completely continuous on $\bar{U}_p$. It is easy to see that $\bar{f}$ is a ∂-mapping on $\bar{U}_p$ satisfying all conditions of the Lemma and, consequently, degree $d[\bar{f}, U, \bar{f}(0)] \neq 0$. Thus, there exists a neighbourhood V of $\bar{f}(0)$ such that V is contained in $\bar{f}(U_p)$. We have, by (2), $\bar{f}(y) = f(p+y)$ for any y of U_p. Thus, image $f(p \oplus U_p)$ contains the neighbourhood V of $f(0) = f(p)$ and Theorem 1 is proved.

The following theorem is a corollary of Theorem 1.

Theorem 2. *Let G be an open set in X and let f be a continuous transformation of the form $f(x) = x - F(x)$, where F is a completely continuous transformation on G with values in X. If f is locally a one-to-one mapping, then image $f(G)$ is an open set in X.*

The following theorem on invariance of domain ([6], [7], [3]) *) is a particular case of Theorem 2.

Theorem 3. *Let G be an open set in X and let f be a continuous transformation of the form $f(x) = x - F(x)$, where F is a completely continuous transformation on G with values in X. If f is a one-to-one mapping, then image $f(G)$ is an open set in X.*

If X is a Banach space, then the following theorem, proved by A. Granas in [5], results from Theorem 1.

Theorem 4. *Let G be an open set in Banach space X. Let us assume that transformation f of the form $f(x) = x - F(x)$, where F is a completely continuous transformation from G into X, is locally an ε-mapping, i. e. f satisfies the following condition: for any x of G there exists a positive number ε such that the closed full sphere $S(x, \varepsilon_x)$ with centre x and radius ε_x is contained in G, and f considered on $S(x, \varepsilon_x)$ is an ε_x-mapping. Then image $f(G)$ is an open set in X.*

2. A continuous transformation f of an open set G in a locally convex linear topological space X into X is strictly locally a ∂-mapping on G, if for every x of G and every neighbourhood $V(x)$ of x such that $V(x) \subset G$ there exists an open convex symmetric neighbourhood U_x of the origin satisfying the following conditions:

1) the closure of $U(x) = x \oplus U_x$ is contained in $V(x)$;

2) the transformation f considered on $\bar{U}(x)$ is a ∂-mapping on $\bar{U}(x)$.

In this section we are concerned with the case when f is an open mapping.

Theorem 5. *Let G be an open set in X and let f be a continuous transformation of the form $f(x) = x - F(x)$, where F is a completely continuous*

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*) A proof of the theorem on invariance of domain in locally convex complete metric space is given in [7]; recently this theorem was proved in [3] for the case of a general locally convex linear topological space, but under various additional restrictions concerning the transformation, see also [8].

transformation on G with values in X. If f is strictly locally a ∂-mapping on G, then f is an open mapping, i. e. f maps every open subset of G into an open set in X.

The proof of this theorem results from that of Theorem 1.

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BULLETIN DE L'ACADÉMIE
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Série des sci. math., astr.
et phys. — Vol. VI, No. 5. 1958

MATHEMATICS

On Shortest Paths on Closed Surfaces

by

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Presented on March 11, 1958

P and Q being two points on a closed and regular surface S, there always exists among all the paths PQ, connecting P with Q on S, one that is shortest. Is it true that for every P on S there exists on S such a Q that there are at least two different paths PQ sharing this advantage of minimum length?

P and Q being given, let us write $d(P, Q)$ for the length of the shortest path on S connecting them; P being fixed and X being a variable point on S, $d(P, X)$ attains its maximum for $X = Q$; we call such a point Q *opposite* to the given P, and it makes no difference, whether there are several opposite points for a given P or not. As to the existence of an opposite point, it is an easy consequence of the continuity of $d(P, X)$ as function of $X \in S$, if we define the distance on S by $d(X, Y)$.

THEOREM. *Let P be arbitrary ($P \in S$) and Q the point opposite to P, as already explained. Among all paths connecting P with Q on S there are two at least of length $d(P, Q)$.*

The theorem above is a positive answer to our query. Some commentaries and preliminary statements are, however, necessary to make the problem clear.

a) We call S *closed and regular* if

1° it is a one-to-one continuous image of the Cartesian sphere

$$x_1^2 + x_2^2 + x_3^2 = 1;$$

2° J being any Jordan curve on S, and D the domain limited by J on S, there exists a set of functions $f_i(u, v)$ $(i = 1, 2, 3)$, continuous for $u^2 + v^2 \leqslant 1$, admitting of continuous partial derivatives up to the third order for $u^2 + v^2 < 1$, such that the formulae

$$x = f_1(u, v), \quad x_2 = f_2(u, v), \quad x_3 = f_3(u, v) \tag{1}$$

convey a one-to-one continuous transformation of the closed domain $u^2 + v^2 \leqslant 1$ into $D + J$, and that the condition $EG - F^2 > 0$ is valid for

$u^2 + v^2 < 1$; the letters E, F, G have the meaning usual in differential geometry. In the sequel, when speaking of S, we always assume for it tacitly properties 1^0 and 2^0, but nothing else.

b) *Path* signifies a simple and rectifiable arc on S or an arc composed of a finite number of such arcs; thus PQR signifies the composite arc $PQ + QR$, both components being simple; we do not, however, exclude the case of the components mutual crossings and overlappings. The *length* of a path is defined as the sum of lengths of its components; l being the length, the path can be defined by $x_i = x_i(s)$ $(i = 1, 2, 3)$ $(0 \leqslant s \leqslant l)$, the functions x indicating the position of a point moving along the path as a function of the length s of the way already covered. The vector $\mathfrak{P}(s) = (x_1(s), x_2(s), x_3(s))$ gives the *description* of the path in question. When speaking of the *tangent* to the path in $\mathfrak{P}(s_0)$ we mean the vector $\dot{\mathfrak{P}}(s_0)$ represented by an arrow issued from $\mathfrak{P}(s_0)$; dots signify d/ds. The length of a tangent is, of course, equal to 1.

We write (PQ) for the length of the path PQ.

c) Let us call *brachistodrome* and denote by $B(P, Q)$ any path PQ for which $(PQ) = d(P, Q)$. $B(P, Q)$ is always a simple arc. Its existence for any pair of points P, Q on S results from the properties of S by the "direct method" due to Hilbert and improved by Lebesgue and Carathéodory [1].

It has been shown [2] that every brachistodrome on S is an *orthodrome*, which means that its description $\mathfrak{P}(s)$ admits of first and second continuous derivatives in $\langle 0, l \rangle$ and satisfies the Lagrange-Eulerian condition for the minimum of length; l designates the length of the brachistodrome in question.

d) It is an almost immediate consequence of the conditions mentioned in c) that the curvature of an orthodrome on S in any of its points, given by $\sqrt{\ddot{x}_1(s)^2 + \ddot{x}_2(s)^2 + \ddot{x}_3(s)^2}$, is equal to the curvature $\varkappa$ of the plane section determined by the normal to S through the point in question, and by the tangent to the orthodrome in the same point. $\varkappa$ is limited on S by a constant k, the same for all points of all orthodromes:

$$\varkappa \leqslant k. \tag{2}$$

e) The existence theorem for the Lagrange-Eulerian differential equations mentioned in c) implies for every orthodrome consisting of a simple arc $PQ\,(P \neq Q)$ the possibility of a prolongation beyond Q to Q' in such a manner that PQ becomes a part of the orthodromic simple arc PQ'.

Following lemmas result easily from (2):

LEMMA 1. *ϑ denoting the angle between the initial and terminal tangent of an orthodrome of length h we have*

$$\vartheta \leqslant kh. \tag{3}$$

Proof. For $0 \leqslant s \leqslant h$ the vector $\dot{\mathcal{P}}(s)$, issued from the point $0,0,0$, traces on the sphere $x_1^2+x_2^2+x_3^2=1$ an auxiliary arc of length

$$\int_0^h \varkappa(s)\,ds\,, \tag{4}$$

where $\varkappa(s)$ is the curvature of the orthodrome in $\mathcal{P}(s)$. The angle ϑ is obviously equal to the angle between $\dot{\mathcal{P}}(0)$ and $\dot{\mathcal{P}}(h)$ which is at most equal to the length of the auxiliary arc; the integral (4) being at most equal to kh because of (2), assertion (3) follows.

Lemma 2. *Let $\theta(s)$ be the angle between the chord $\overrightarrow{PQ}$ connecting the initial point P of an orthodrome of length h with its terminal point Q and an arbitrary tangent to said orthodrome; we have*

$$\theta(s) \leqslant kh\,. \tag{5}$$

Proof. As θ is at most equal to the maximum of the angle between $\mathcal{T}(s_1)$ and $\mathcal{T}(s_2)$ for $0 \leqslant s_1 \leqslant s_2 \leqslant h$ — we denote here and in the sequel by $\mathcal{T}(s)$ the tangent in $\mathcal{P}(s)$ — lemma 1 implies (5).

Lemma 3. *Same notations as above. Calling c the length of the chord of lemma 2 we get*

$$h\left(l-\frac{k^2h^2}{2}\right) \leqslant c \leqslant h\,. \tag{6}$$

Proof. 1° We have $c \geqslant h\cdot\cos\gamma$, γ being the maximum of the angles between chord and $\mathcal{T}(s)$ for $0 \leqslant s \leqslant h$. 2° By lemma 2 $\gamma \leqslant kh$, thus $\cos\gamma \geqslant l-k^2h^2/2$. 1° and 2° imply (6).

The two lemmas which follow concern sequences of paths.

Lemma 4. *Suppose $B(P,Q)$ to be the unique brachistodrome connecting P with Q and call l its length. Let $\{P_nQ_n\}$ be a sequence of paths and* $\lim P_n = P$, $\lim Q_n = Q$, $(P_nQ_n) = l$. *Let us write $\mathcal{D}(s)$ for the description of PQ and $\mathcal{D}_n(s)$ for such of P_nQ_n. Then*

$$\lim \mathcal{D}_n(s) = \mathcal{D}(s) \quad \text{uniformly in} \quad \langle 0, l\rangle\,. \tag{7}$$

Proof. All the components of $\mathcal{D}_n(s)$ being uniformly bounded, and their variations being at most l, there exists an uniformly convergent subsequence $\{\mathcal{D}_{n_j}(s)\}$; its limit defines a trajectory of length at most l, connecting P with Q. Because of the uniqueness of $B(P,Q)$ the trajectory is identical with $B(P,Q)$ and $\lim\limits_{j\to\infty} \mathcal{D}_{n_j}(s) = \mathcal{D}(s)$. Thus, all subsequences of $\mathcal{D}_n(s)$ contain subsequences of their own converging uniformly towards $\mathcal{D}(s)$, which implies (7).

Lemma 5. *Notations as in preceding lemma. If the conditions of lemma 4 are satisfied, and if every path P_nQ_n is a simple arc P_nQ+QQ_n, the part P_nQ being an orthodrome, we get*

$$\lim \dot{\mathcal{D}}_n(0) = \dot{\mathcal{D}}(0)\,. \tag{8}$$

Proof. For $0 < h < l$ the points $\mathfrak{D}_n(h)$ belong, for $n > N(h)$, to P_nQ. h being fixed we get

$$\lim \mathfrak{D}_n(0) = \mathfrak{D}(0), \quad \lim \mathfrak{D}_n(h) = \mathfrak{D}(h)$$

which shows the convergence of the chord $\mathfrak{D}_n(0)\,\mathfrak{D}_n(h)$ towards the chord $\mathfrak{D}(0)\,\mathfrak{D}(h)$. All these chords being for $n > N(h)$ chords of orthodromes we apply lemma 2 to the angles between the initial tangent $\mathfrak{T}_n(0)$ to the orthodromes and the corresponding chords; (5) shows that no such angle exceeds kh. The same being true for the angle between $\mathfrak{T}(0)$ and $\mathfrak{D}(0)\,\mathfrak{D}(h)$, we get

(9) $$\limsup |\dot{\mathfrak{D}}_n(0) - \dot{\mathfrak{D}}(0)| \leqslant 2kh\,.$$

Since h is arbitrarily small, (9) implies (8).

Proof. Fig. 1 helps to follow the essential part of our argument.

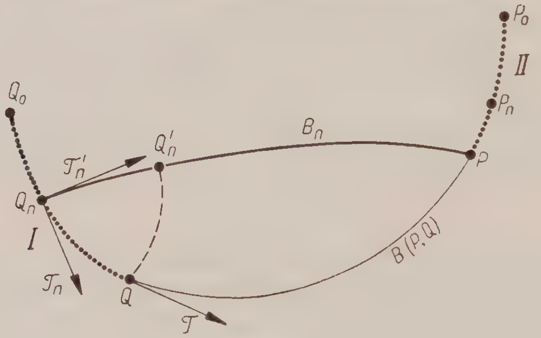

Fig. 1

P being given, let Q be opposite to P and assume the uniqueness of $B(P, Q)$, against the assertion of our theorem. Applying e) to both endpoints of $B(P, Q)$ we draw prolongations I and II beyond Q, respectively beyond P; we determine two points, Q_0 and P_0, on I and II respectively, so as to have $(Q_0Q) = (PP_0)$; paths symbolized by terminal points only are orthodromes here and in the sequel. Let Q_n be a point lying on I between Q_0 and Q. Calling B_n the brachistodrome $B(Q_n, P)$, B the brachistodrome $B(P, Q)$, and l_n, l their respective lengths we get

(10) $$l_n \leqslant l > 0, \quad \text{and} \quad (QQ_n) + l_n \geqslant l;$$

the first inequalities result from Q being opposite to P, the last from B being a brachistodrome. They imply the existence of a point P_n on PP_0 such that

(11) $$l_n + (PP_n) = l, \quad \text{and} \quad (PP_n) \leqslant (QQ_n) \leqslant (QQ_0)\,.$$

Let a new point $Q'_n \in B_n$ be defined by

(12) $$h'_n = (Q_nQ'_n) = (Q_nQ) = h_n$$

giving rise to a new brachistodrome QQ'_n.

Let us now denote the path $B_n + PP_n$ by $Q_n P_n$ and suppose $\lim (Q_n Q) = 0$, considering n as an index $(n = 1, 2, \ldots)$. This implies $\lim (P_n P) = 0$ in view of (11) and permits to apply lemma 5 to the sequence $\{Q_n P_n\}$ with the result that

(13) $$\lim \mathfrak{T}'_n = \mathfrak{T},$$

$\mathfrak{T}'_n$ denoting the tangent to $Q_n P_n$ in Q_n and $\mathfrak{T}$ the tangent to B in Q. Calling $\mathfrak{T}_n$ the tangent to $Q_n Q$ in Q_n we get obviously

(14) $$\lim \mathfrak{T}_n = \mathfrak{T};$$

(13) and (14) yield

(15) $$\lim \beta_n = 0$$

for the angle β_n at the top Q_n of the isosceles orthodromic triangle $\Delta = QQ_nQ'_n$ drawn of Fig. 2.

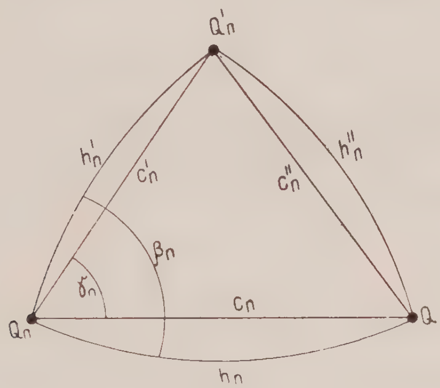

Fig. 2

Because of lemma 2 and

(16) $$\lim h'_n = \lim h_n = 0$$

(15) implies for the angle γ_n at the vertex Q_n of the ordinary triangle $D = \overline{QQ_nQ'_n}$

(17) $$\lim \gamma_n = 0.$$

Q'_nQ being a brachistodrome we get

$$h''_n = (Q'_nQ) \leqslant h'_n + h_n = 2h_n,$$

thus, in view of (16),

(18) $$\lim h''_n = 0.$$

Let us apply lemma 2 putting in (6) successively h_n, h'_n and h''_n for h, and c_n, c'_n and c'', respectively, for c — we get, because of (16) and (18),

(19) $$\lim c_n/h_n = \lim c'_n/h'_n = \lim c''_n/h''_n = 1.$$

(12) and (19) imply

$$\lim c_n'/c_n = 1 \tag{20}$$

which with (17) leads to

$$\lim c_n''/c_n' = 0\,. \tag{21}$$

Combining (19) with (20) and (21) we get $\lim h_n''/h_n' = 0$, which gives

$$h_n' > h_n'' \tag{22}$$

for an appropriate n; for such an n (11) and (22) imply

$$l \geqslant (PQ_n) = (PQ_n') + (Q_n'Q_n) > (PQ_n') + (Q_n'Q)\,,$$

and, consequently, the path $PQ_n'Q$ is shorter than $B(P, Q)$, against the definition of $B(P, Q)$. This contradiction has been deduced from the assumption of the uniqueness of $B(P, Q)$ — thus the Theorem is true.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 5. 1958

MATHEMATICS

A Remark Concerning the Multiplicative Linear Functionals

by

S. MRÓWKA

Presented by S. MAZUR on March 7, 1958

Suppose K is a field and let R_t $(t \in T)$ be linear algebras over K, each having the unit element e_t. Denote by R the product $\underset{t \in T}{P} R_t$ of algebras R_t. A. Białynicki-Birula and W. Żelazko have proved [1] *) the following theorems (K is supposed to be an infinite field):

(A): If $\overline{\overline{T}} < \aleph_I$, then every multiplicative linear functional f on R is given by

$$(*) \qquad f(x) = f_{t_0}(x(t_0)) \quad (x \in R),$$

where t_0 is a fixed element of T and f_{t_0} is a fixed multiplicative linear functional on R_{t_0}.

(B): If $\overline{\overline{T}} \geqslant \aleph_I$ and $\overline{\overline{K}} < \aleph_I$, then there exists a family R_t, $t \in T$, such that there exists a multiplicative linear functional f defined over R which cannot be written in the form (*)

(C): If $\overline{\overline{K}} \geqslant \overline{\overline{T}}$, then every multiplicative linear functional f defined on R may be written in the form (*).

($\aleph_I$ has the same meaning as in [1]).

In this paper we give a generalization of the above theorems. We do not assume that K is an infinite field.

Suppose T is a non-void set. By an $\mathfrak{m}$-*additive measure over* T ($\mathfrak{m}$ is a cardinal) we understand, in this paper, a function μ defined on 2^T (= the class of all subsets of T) which takes values 0 and 1 and

*) The present paper is a supplement to paper [1] the knowledge of which is here assumed.

satisfies the following conditions:

1^0 $\mu(T) = 1$; $\mu(\emptyset) = 0$; *)

2^0 $\mu(A) \leqslant \mu(B)$ for $A \subset B \subset T$;

3^0 if $\mu(A) = 1$, then $\mu(T \setminus A) = 0$;

4^0 $\mu(\bigcup\{A: A \in \mathfrak{A}\}) = \sup\{\mu(A): A \in \mathfrak{A}\}$ for each $\mathfrak{A} \subset 2^T$ with $\overline{\overline{\mathfrak{A}}} \leqslant \mathfrak{m}$.

An $\mathfrak{m}$-additive measure μ over T is said to be *atomic* provided that there is a point $t_0 \in T$ with $\mu(\{t_0\}) = 1$. For each cardinal $\mathfrak{m}$ we denote as $\aleph(\mathfrak{m})$ the least cardinal $\mathfrak{n}$ for which there exists a set T of the power $\mathfrak{n}$ and a non-atomic $\mathfrak{m}$-additive measure over T.

Let us notice that:

(i) $\mathfrak{m} < \aleph(\mathfrak{m})$;

(ii) $\aleph(\aleph_0) = \aleph_1$;

(iii) $\aleph(2) = \aleph(3) = \ldots = \aleph_0$.

Only (iii) needs a proof. Clearly, $\aleph(2) = \aleph(3) = \ldots \geqslant \aleph_0$. Conversely, let T be any set of the power $\aleph_0$ and let I be a maximal proper ideal of subsets of T which contains all finite subsets of T. Setting $\mu(A) = 0$ for $A \in I$ and $\mu(A) = 1$ for $A \bar{\in} I$, we obtain a non-atomic 2-additive measure over T. Thus, $\aleph(2) \leqslant \aleph_0$ and (iii) is proved.

We shall show the following

THEOREM 1. *If $\overline{\overline{T}} < \aleph(\overline{\overline{K}})$, then each multiplicative linear functional f defined on R is of the form* (*).

THEOREM 2. *If, for some cardinal $\mathfrak{p}$, $\overline{\overline{K}} < \aleph(\mathfrak{p})$ and $\overline{\overline{T}} \geqslant \aleph(\mathfrak{p})$, then there exists a family R_t, $t \in T$, such that there exists a multiplicative linear functional f defined on R which cannot be written in the form* (*).

We start out with the proof of Theorem 1 **). Let 0 and 0_t denote the zero element of K and R_t, respectively, and let χ_A be the characteristic function of a set $A \subset T$ (i. e. $\chi_A(t) = e_t$ for $t \in A$ and $\chi_A(t) = 0_t$ for $t \in T \setminus A$). Then χ_T is the unit element of R. Clearly, one can assume that the functional f does not vanish identically on R; then $f(\chi_T) = e$ (e denotes the unit element of the field K).

Denote as $\mathfrak{m}$ the power of the field K. Let us set for $A \subset T$:

$$\mu(A) = 0 \quad \text{if} \quad f(\chi_A) = 0;$$
$$\mu(A) = 1 \quad \text{in the opposite case.}$$

We are going to show that μ is an $\mathfrak{m}$-additive measure over T. Clearly, the conditions 1^0-3^0 are satisfied; on the other hand, it is plain that μ is finitely additive, thus one can assume that $\mathfrak{m}$ is infinite. Let $\mathfrak{A} = \{A_\xi: \xi \in \Xi\}$ $(\overline{\overline{\Xi}} \leqslant \mathfrak{m})$ be a family of subsets of T and suppose that $\mu(A_\xi) = 0$ for each ξ in Ξ. There exists a family $\mathfrak{C} = \{C_\xi: \xi \in \Xi\}$ such that

*) $\emptyset$ denotes the empty set.

**) The idea of this proof is the same as that of the proof of (A) given in [1].

$AC_\xi \subset A_\xi$, $C_\xi \cap C_{\xi'} = \emptyset$ for $\xi, \xi' \in \Xi$, $\xi \neq \xi'$, and $B = \bigcup \{A_\xi; \xi \in \Xi\} = \bigcup \{C_\xi; \xi \in \Xi\}$.

To each $\xi \in \Xi$ an element $a_\xi \neq 0$ of the field K can be associated in such a way that $a_\xi \neq a_{\xi'}$ for $\xi \neq \xi'$. Let us set

$$z(t) = a_\xi \cdot e_t \quad \text{for} \quad t \in C_\xi;$$
$$z(t) = 0_t \quad \text{for} \quad t \in T \setminus B.$$

We shall show that $f(z) = 0$. If $f(z) = a \neq 0$, then $f(u) = 0$, where $u = z - a \cdot \chi_T$. If $a \neq a_\xi$ for each ξ in Ξ, then u has an inverse in R and it follows $f(u) \neq 0$. Hence $a = a_{\xi_0}$ for some ξ_0 in Ξ. Then $f(u - \chi_{C_{\xi_0}}) = 0$, but $u - \chi_{C_{\xi_0}}$ has an inverse and it leads to a contradiction. Finally, $f(z) = 0$. Setting

$$z_1(t) = \frac{1}{a_\xi} \cdot e_t \quad \text{for} \quad t \in C_\xi;$$
$$z_1(t) = 0_t \quad \text{for} \quad t \in T \setminus B$$

we have $z \cdot z_1 = \chi_B$, hence $f(\chi_B) = 0$ and $\mu(B) = 0$. Thus, the condition 4° is also satisfied and μ is an $\mathfrak{m}$-additive measure over T. Since $\overline{\overline{T}} < \aleph(\mathfrak{m})$, there exists an element $t_0 \in T$ such that $\mu(\{t_0\}) = 1$. Then, it may be shown in an analogous manner as in [1] that for some functional f_{t_0} on R_{t_0} we have

$$f(x) = f_{t_0}(x(t_0)) \quad \text{for each } x \text{ in } R.$$

Thus Theorem 1 is proved.

The proof of Theorem 2 is the same as that of (B) given in [1], and it may be left to the reader.

Notice that (A) and (C) follow from Theorem 1 by (i) and (ii), respectively, and (B) follows from Theorem 2 by (i). In virtue of (iii) we obtain the following statement concerning finite fields:

Suppose K is a finite field. If T is finite, then each multiplicative linear functional f on R is of the form (*). *If T is infinite, then there exists a family R_t, $t \in T$, such that there exists a multiplicative linear functional f on R being not of the form* (*).

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 5. 1958

MATHEMATICS

On Subspaces of a Space with an Absolute Basis

by

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Presented by S. MAZUR on March 2, 1958

In this note we shall generalize the results obtained by R. C. James in [4] and [5] to the case of a subspace of a space with an absolute basis *).

1. Notation. The terminology and notation used in this note are the same as in [2] and [6]. In the sequel we shall always denote by X the space with the absolute basis (x_n) and with the biorthogonal system (x_n, f^n), by Y — a subspace of the space X. If $f \in X^*$, then we shall denote by f_Y the functional belonging to Y^* such that $f_Y(y) = f(y)$, for every $y \in Y$.

2. LEMMA 1. *If $f \in X^*$, then $f(x) = \sum_{n=1}^{\infty} f(x_n) f^n(x)$ for every $x \in X$ and the series $\sum_{n=1}^{\infty} f(x) f^n$ is weakly unconditionally convergent*

We shall omit the simple proof.

LEMMA. 2. *If $g \in Y^*$, and f is an extension of g to the space X, i. e. $g = f_Y$, then $g(y) = \sum_{n=1}^{\infty} f(x_n) f_Y^n(y)$ for every $y \in Y$ and the series $\sum_{n=1}^{\infty} f(x_n) f_Y^n$ is weakly unconditionally convergent.*

This lemma is a trivial consequence of Lemma 1.

THEOREM 1. *The following conditions are equivalent*:

(11) *in the space Y every bounded set is weakly conditionally compact*;

(12) *Y^* is weakly complete*;

(13) *no subspace of Y^* is isomorphic to c_0*;

(14) *no subspace at Y isomorphic to l is complemented in Y*;

(15) *if Y_1 is a separable subspace of Y, then Y_1^* is separable*;

(16) *Y contains no subspace isomorphic to l.*

*) A considerable part of the results given here, was obtained realier by the authors in another way, see [7].

Proof. (11)→(12). According to Corollary 2 of [6], the space Y has the property (u). Now we note that the space Y fulfills the assumptions of Corollary 5 of [6].

(12)→(13). This implication is trivial.

(13)→(14). This implication follows immediately from Theorem 1 of [2].

(14)→(15). From (14), according to Corollary 1 of [2], it follows that in Y_1 every weakly unconditionally convergent series is convergent. Thus, it follows from Lemma 2 that the set of all linear combinations of elements $f^n_{Y_1}$ $(n = 1, 2, \ldots)$ is dense in Y_1^*; hence Y_1^* is separable.

(15)→(11). This fact is well-known [1].

The implications "(11)→(16)→(14)" are trivial.

Theorem 2. *The space Y is weakly complete, if and only if, no subspace of Y is isomorphic to c_0.*

This theorem follows immediately from Corollary 2 and Theorem 1 of [6].

Theorem 3. *The following conditions are equivalent*

(31) *Y is reflexive,*

(32) *every bounded set in Y^* is conditionally weakly compact,*

(33) *no subspace of Y^* is isomorphic to l.*

(34) *no subspace of Y is isomorphic either to l or to c_0.*

Proof. (31)→(32). If Y is reflexive then Y^* is reflexive also, hence, every bounded set in Y^* is conditionally weakly compact.

(32)→(33). This implication is trivial.

(33)→(34). If Y contains a subspace isomorphic to l then, according to Theorem 1, Y contains a subspace isomorphic to l and complemented in Y, hence, according to Theorem 1 of [2], Y contains a subspace isomorphic to m. Since $m \supset l$, Y contains a subspace isomorphic to l. If Y contains a subspace isomorphic to c_0 then, according to Corollary 2 of [2], Y contains a subspace isomorphic to l.

(34)→(31). It follows from Theorems 1 and 2 that every bounded set in Y is conditionally weakly compact and Y is weakly complete. Therefore, every bounded set in Y is sequentially weakly compact (with respect to Y) and, according to the result of Eberlain [3], Y is reflexive.

Definition. Let $\mathfrak{A}$ be an abstract set of indices α. The set (x_α), where $\alpha \in \mathfrak{A}$ is called an *absolute basis of the power* $\overline{\overline{\mathfrak{A}}}$ if the set of all linear combinations of elements x_α $(\alpha \in \mathfrak{A})$ is dense in X and every countable subset of (x_α) composes and absolute basis of a subspace of X.

Remark. Theorems 1-3 remain true if we replace the assumption "X possesses an absolute basis" by the assumption "X possesses an absolute basis of power $\overline{\overline{\mathfrak{A}}}$".

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POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 5. 1958

MATHÉMATIQUE

Sur les extrêmes des fonctions composées par des intégrales des équations aux dérivées partielles du type elliptique

par

B. PIŁAT

Présenté par T. WAŻEWSKI, le 17 mars, 1958

Dans le travail [2] M. Biernacki a énoncé et démontré, entre autres, le théorème suivant dans le cas où $a_{ik} \equiv 0$ pour $i \neq k$;

THÉORÈME. *Soit*

$$\Delta^* u \equiv \sum_{i,k=1}^{n} a_{ik}(x_1, \ldots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x_1, \ldots, x_n) \frac{\partial u}{\partial x_i}$$

où les coefficients $a_{ik}(x_1, \ldots, x_n)$, $b_i(x_1, \ldots, x_n)$ *sont continus et bornés dans un domaine* D, *la forme* $\sum_{i,k=1}^{n} a_{ik}\lambda_i\lambda_k$ *est définie positive et son déterminant est supérieur à une constante positive dans ce domaine. Si* $g(x_1, \ldots, x_n)$ *et* $h_\nu(x_1, \ldots, x_n)$, $\nu = 1, 2, \ldots, s$, *sont des fonctions de la classe* $C^{(2)}$ *dans* D *qui satisfont dans* D *aux inégalités* $\Delta^* g \leqslant 0$ *et* $\operatorname{sign} h_\nu \cdot \Delta^* h_\nu \leqslant 0$ *alors la fonction*

$$w(x_1, \ldots, x_n) = e^{g(x_1, \ldots, x_n)} h_1(x_1, \ldots, x_n) \ldots h_s(x_1, \ldots, x_n)$$

ne peut avoir en un point P *intérieur à* D *ni un minimum relatif positif* w_1 *ni un maximum relatif négatif* w_2, *à moins qu'elle ne se réduise à une constante dans tout domaine* D', *contenant le point* P *et contenu dans* D, *où l'on a* $w \geqslant w_1$ *ou* $w \leqslant w_2$. *Si, en plus, on a dans* D *l'inégalité* $\Delta^* g < 0$ *ou* $\operatorname{sign} h_\nu \cdot \Delta^* h_\nu < 0$ *pour un* $\nu (\nu = 1, 2, \ldots, s)$, *alors* $w(x_1, \ldots, x_n)$ *n'admet dans* D *ni un minimum relatif positif, ni un maximum relatif négatif.*

Je vais démontrer le théorème dans le cas général, en profitant du théorème suivant, dû à G. Ascoli [1] et E. Hopf [3] (cf. aussi C. Miranda [4]):

"Considerons l'expression

$$L(u) \equiv \sum_{i,k=1}^{n} a_{ik}(x_1, \ldots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x_1, \ldots, x_n) \frac{\partial u}{\partial x_i} + c(x_1, \ldots, x_n) u$$

où tous les coefficients a_{ik}, b_i, c sont continus et bornés dans un domaine D, et $c \leqslant 0$ tandis que la forme quadratique $\sum_{i,k=1}^{n} a_{ik} \lambda_i \lambda_k$ est définie positive dans D, de manière que son déterminant y reste supérieur à un nombre positif fixe. Aucune solution de l'équation $L(u) = 0$ qui est de classe $C^{(2)}$ dans D ne peut avoir en un point P intérieur à D un maximum relatif positif u_1 (un minimum relatif négatif u_2) à moins qu'elle ne se réduise à une constante dans tout le domaine D' contenant P et contenu dans D où l'on a $u \leqslant u_1$ $(u \geqslant u_2)$. Si $c(x_1, \ldots, x_n) < 0$ dans D, la dernière circonstance ne se présente pas".

Considérons maintenant la fonction:

$$u(x_1, \ldots, x_n) = w^{-1}(x_1, \ldots, x_n) = e^{-g(x_1, \ldots, x_n)} [h_1(x_1, \ldots, x_n) \ldots h_s(x_1, \ldots, x_n)]^{-1} .$$

On a

$$\frac{\partial u}{\partial x_i} = -u \left(g_{x_i} + \sum_{\nu=1}^{s} \frac{h_{\nu x_i}}{h_\nu} \right) = -u\lambda_i$$

et

$$\frac{\partial^2 u}{\partial x_i \partial x_k} = u\lambda_i \lambda_k - u \left(g_{x_i x_k} + \sum_{\nu=1}^{s} \frac{h_{\nu x_i x_k}}{h_\nu} - \sum_{\nu=1}^{s} \frac{h_{\nu x_i} h_{\nu x_k}}{h_\nu^2} \right)$$

donc

$$(*) \qquad \frac{\Delta^* u}{u} =$$

$$= \sum_{i,k=1}^{n} a_{ik} \lambda_i \lambda_k + \sum_{\nu=1}^{s} \frac{1}{h_\nu^2} \sum_{i,k=1}^{n} a_{ik} h_{\nu x_i} h_{\nu x_k} - \sum_{\nu=1}^{s} \frac{\Delta^* h_\nu}{h_\nu} - \Delta^* g = -c(x_1, \ldots, x_n) .$$

L'expression $\dfrac{\Delta^* u}{u}$ est non négative dans D, puisque les deux premiers termes dans $(*)$ sont composés des valeurs de la forme quadratique positive considérée et l'on a $\operatorname{sign} h_\nu \cdot \Delta^* h_\nu \leqslant 0$ et $\Delta^* g \leqslant 0$; dans le cas où l'on a, dans D, l'inégalité $\Delta^* g < 0$ ou $\operatorname{sign} h_\nu \cdot \Delta^* h_\nu < 0$ pour un $\nu (\nu = 1, 2, \ldots, s)$, l'expression $\dfrac{\Delta^* u}{u}$ est positive dans D. De là et du théorème cité résulte la proposition énoncée au début.

Dans le travail [2] M. Biernacki a aussi posé la question suivante: *dans le cas où $\Delta^* = \Delta$ est l'opérateur de Laplace, une somme d'expressions du type $e^g h_1 \ldots h_s$ de même signe a-t-elle toujours la même propriété que ses termes?*

L'exemple suivant montre *que la réponse est négative*: soit $h = 4-x$, $g = 4x$, $1 \leqslant x \leqslant 3$. Les fonctions h et g sont harmoniques et positives dans cet intervalle, alors que l'expression

$$s(x) = h^2(x) + g(x) = (4-x)^2 + 4x = (x-2)^2 + 12$$

admet pour $x = 2$ un minimum positif.

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et phys. — Vol. VI, No. 5. 1958

MATHÉMATIQUE

Sur un problème de Mlle Z. Szmydt relatif à l'équation $\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$

par

A. BIELECKI et J. KISYŃSKI

Présenté par T. WAŻEWSKI le 17 mars 1958

1. Hypothèses. Désignons par Δ le rectangle: $|x| \leqslant \alpha$, $|y| \leqslant \beta$ où $\alpha > 0$ et $\beta > 0$, et par D le domaine: $(x, y) \in \Delta$, z, p, q arbitraires. Soit $f(x, y, z, p, q)$ une fonction continue et bornée dans D, satisfaisant dans D à la condition

$$(1) \qquad |f(x, y, z, p, q) - f(x, y, z, \bar{p}, \bar{q})| \leqslant \varphi(|p-\bar{p}| + |q-\bar{q}|),$$

où $\varphi(\delta)$ est une fonction continue et non-décroissante pour $\delta \geqslant 0$, telle que $\varphi(0) = 0$ et

$$(2) \qquad \int_0^\delta \frac{du}{\varphi(u)} = +\infty, \quad \text{pour} \quad \delta > 0.$$

Nous admettons de plus que $G(x, z, q)$ et $H(y, z, p)$ sont des fonctions continues et bornées pour $|x| \leqslant \alpha$ et $|y| \leqslant \beta$ qui remplissent les conditions:

$$|G(x, z, q) - G(x, z, \bar{q})| \leqslant A \cdot |q-\bar{q}|, \qquad |H(y, z, p) - H(y, z, \bar{p})| \leqslant B|p-\bar{p}|,$$

où $A > 0$, $B > 0$ et $AB < 1$. Enfin, nous admettons que $g(x)$ et $h(y)$ sont des fonctions définies et continues pour $|x| \leqslant \alpha$ resp. $|y| \leqslant \beta$, telles que $g(0) = h(0) = 0$, $|g(x)| \leqslant \beta$, $|h(y)| \leqslant \alpha$, $|g(x) - g(\bar{x})| \leqslant a \cdot |x - \bar{x}|$, $|h(y) - h(\bar{y})| \leqslant b \cdot |y - \bar{y}|$, où $a > 0$, $b > 0$, $ab < 1$, et que $\mathring{x}$, $\mathring{y}$ et $\mathring{z}$ sont trois nombres satisfaisant aux inégalités $|\mathring{x}| \leqslant \alpha$ et $|\mathring{y}| \leqslant \beta$.

2. Théorème. *Dans les hypothèses qui viennent d'être énoncées il existe une fonction $z = z(x, y)$ continue, ayant des dérivées $\partial z/\partial x$, $\partial z/\partial y$, $\partial^2 z/\partial x \partial y$ continues dans Δ et satisfaisant à l'équation différentielle*

$$(3) \qquad \frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \quad \textit{dans } \Delta$$

et aux conditions aux limites

$$\frac{\partial z}{\partial x} = G\left(x, z, \frac{\partial z}{\partial y}\right), \quad \textit{pour} \quad |x| \leqslant a,\ y = g(x), \tag{4}$$

$$\frac{\partial z}{\partial y} = H\left(y, z, \frac{\partial z}{\partial x}\right), \quad \textit{pour} \quad |y| \leqslant \beta,\ x = h(y), \tag{5}$$

et

$$z(\mathring{x}, \mathring{y}) = \mathring{z}. \tag{6}$$

Le problème de l'existence d'une solution de l'équation (3) assujettie aux conditions (4)-(6) a été posé par Mlle Z. Szmydt ([2], p. 68, Problème I) et résolu positivement par elle-même ([2], p. 69, Théorème 1) dans le cas spécial où les fonctions G et H ne dépendent que des variables x resp. y, mais en admettant une condition plus faible que (1) et (2) *).

Un problème semblable, encore plus général, a été envisagé par Mlle Z. Szmydt dans un autre travail ([3], p. 580, Problème I*) dans l'hypothèse que la fonction $f(x, y, z, p, q)$ satisfait à la condition de Lipschitz par rapport à p et q ([3], p. 582, Théorème 2). La méthode utilisée dans [3] ne s'applique plus ici **) et nous allons énoncer un lemme qui nous permettra de surmonter les difficultés qui se présentent dans la démonstration de notre théorème.

3. Lemme. *Supposons que les fonctions continues $g(x)$ et $h(x)$ remplissent, pour $|x| \leqslant a$ resp. $|y| \leqslant \beta$ les inégalités $|g(x)| \leqslant \min(\beta, a|x|)$ et $|h(y)| \leqslant \min(a, b|y|)$, où $a > 0$, $b > 0$, $ab < 1$, et soient $\varphi^*(\delta)$ et $\Omega^*(\delta)$ des fonctions continues et non-décroissantes pour $\delta \geqslant 0$ et satisfaisant aux conditions $\varphi^*(0) = \Omega^*(0) = 0$, $\varphi^*(\delta) \leqslant$ const et à la condition analogue à (2). Admettons que*

$$g^1(x) = g(x), \qquad h^1(x) = h(x),$$

$$g^{n+1}(x) = g\big(h(g^n(x))\big),\ h^{n+1}(y) = h\big(g(h^n(y))\big) \quad \textit{pour} \quad n = 1, 2, \ldots,$$

$$\lambda(x, y) = |y - g(x)| + |x - h(g(x))| + \sum_{n=1}^{\infty} \Big\{ |g^n(x) - g^{n+1}(x)| + |h(g^n(x)) - h(g^{n+1}(x))| \Big\}, \tag{7}$$

$$\mu(x, y) = |x - h(y)| + |y - g(h(y))| + \sum_{n=1}^{\infty} \Big\{ |h^n(y) - h^{n+1}(y)| + |g(h^n(y)) - g(h^{n+1}(y))| \Big\} \tag{8}$$

---

*) Il y était question d'un système de n équations du type (3) à n fonctions inconnues. Cependant, pour éviter des complications qui ne sont pas essentielles, nous nous contenterons, dans cette note concise, d'une seule équation.

**) Notre condition exprimée par les formules (1) et (2), moins restrictive que celle de Lipschitz, mais plus forte que celle de Mlle Z. Szmydt ([2], p. 69, Hypothèse K), a été déjà employée par J. Kisyński dans son travail consacré à l'étude des problèmes généralisés de Cauchy et de Goursat [1].

et $\varepsilon_1^*(x,y,\delta)=\eta(\lambda(x,y),\delta)$, $\varepsilon_2^*(x,y,\delta)=\eta(\mu(x,y),\delta)$, *où* $\eta(s,\delta)$ *désigne une solution de l'équation* *)

(9) $$\eta(s,\delta)=(1-\theta)^{-1}\Omega^*(\delta)+\int_0^s\varphi^*\big(\eta(\sigma,\delta)\big)d\sigma,$$

θ *est une constante,* $0<\theta<1$, $(x,y)\in\Delta$, $s\geqslant 0$ *et* $\delta\geqslant 0$.

Dans ces hypothèses les fonctions $\varepsilon_1^*(x,y,\delta)$ et $\varepsilon_2^*(x,y,\delta)$ *sont nonnégatives, continues et non-décroissantes par rapport à* δ,

(10) $$\Omega^*(\delta)+\theta\cdot\varepsilon_2^*(x,g(x),\delta)+\left|\int_{g(x)}^{y}\varphi^*\big(\varepsilon_1^*(x,t,\delta)\big)dt\right|\leqslant\varepsilon_1^*(x,y,\delta),$$

(11) $$\Omega^*(\delta)+\theta\cdot\varepsilon_1^*\big(h(y),y,\delta\big)+\left|\int_{h(y)}^{x}\varphi^*\big(\varepsilon_2^*(s,y,\delta)\big)ds\right|\leqslant\varepsilon_2^*(x,y,\delta),$$

pour $(x,y)\in\Delta$ *et* $\delta\geqslant 0$, *et on a*

(12) $$\lim_{\delta\to 0+}\{\max_{(x,y)\in\Delta}\varepsilon_i^*(x,y,\delta)\}=0\qquad pour\quad i=1,2.$$

4. Démonstration du lemme. L'inégalité $ab<1$ assure la convergence uniforme des séries (7) et (8) et la continuité des fonctions $\lambda(x,y)$ et $\mu(x,y)$. La fonction $\eta(s,\delta)$ étant évidemment continue et non-décroissante par rapport à δ et $\eta(s,0)\equiv 0$, on en déduit facilement (12). Enfin, on constate que

$$\lambda\big(x,g(x)\big)=\mu\big(x,g(x)\big),\qquad\lambda\big(h(y),y\big)=\mu\big(h(y),y\big)$$

et

$$\eta(s,\delta)\geqslant(1-\theta)^{-1}\Omega^*(\delta),$$

d'où

$$\varepsilon_1^*\big(x,g(x),\delta\big)\geqslant\Omega^*(\delta)+\theta\varepsilon_2^*\big(x,g(x),\delta\big),\ \ \varepsilon_2^*\big(h(y),y,\delta\big)\geqslant\Omega^*(\delta)+\theta\varepsilon_1^*\big(h(y),y,\delta\big);$$

pour en obtenir les inégalités (10) et (11) il suffit d'appliquer les identités

$$\varepsilon_1^*\big(x,g(x),\delta\big)+\left|\int_{g(x)}^{y}\varphi^*\big(\varepsilon_1^*(x,t,\delta)\big)dt\right|=\varepsilon_1^*(x,y,\delta),$$

$$\varepsilon_2^*\big(h(y),y,\delta\big)+\left|\int_{h(y)}^{x}\varphi^*\big(\varepsilon_2^*(s,y,\delta)\big)ds\right|=\varepsilon_2^*(x,y,\delta)$$

que l'on vérifie par un calcul assez simple.

5. Démonstration du théorème. Soit

$$M\geqslant\max\{|f(x,y,z,p,q)|,\ |G(x,z,q)|,\ |H(y,z,p)|\}$$

*) Cette équation admet, dans l'intervalle $0\leqslant s<+\infty$ une, et une seule solution pour tout $\delta\geqslant 0$, ce qui résulte de l'équation (2).

pour $(x, y) \in \Delta$,

$$P = (2\beta+1)M, \quad Q = (2\alpha+1)M, \quad Z = |\mathring{z}|+2(\alpha P+\beta Q), \quad K = P+2\beta M+Q$$

et soit $\omega(\delta)$ le module de continuité pour les fonctions f, G et H considérées dans le domaine: $(x, y) \in \Delta$, $|z| \leqslant Z$, $|p| \leqslant P$, $|q| \leqslant Q$ *). Posons

$$\bar{\varphi}(u) = \min\{\varphi(u), 2M\}, \qquad \varphi^*(u) = \left(\frac{1}{\sqrt{A}}+\frac{1}{\sqrt{B}}\right)\bar{\varphi}\big((\sqrt{A}+\sqrt{B})u\big),$$

$$\Omega(\delta) = M(a+b)\delta+(2\alpha+2\beta+1)\cdot\omega\big([1+K(1+a+b)+M]\delta\big),$$

$$\Omega^*(\delta) = \left(\frac{1}{\sqrt{A}}+\frac{1}{\sqrt{B}}\right)\cdot\Omega\left(\left[\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right]\delta\right), \qquad \theta = \sqrt{AB},$$

$$\varepsilon_1(x, y, \delta) = \sqrt{A}\varepsilon_1^*(x, y, \sqrt{a}\delta), \qquad \varepsilon_2(x, y, \delta) = \sqrt{B}\varepsilon_2^*(x, y, \sqrt{b}\delta)$$

et désignons par W l'ensemble de tous les couples $\{p(x, y), q(x, y)\}$ de fonctions définies et continues dans Δ, telles que

$$|p(x, y)| \leqslant P, \qquad |q(x, y)| \leqslant Q,$$
$$|p(x, y)-p(x, \bar{y})| \leqslant M\cdot|y-\bar{y}|, \qquad |q(x, y)-q(\bar{x}, y)| \leqslant M\cdot|x-\bar{x}|,$$
$$|p(x, y)-p(\bar{x}, y)| \leqslant \varepsilon_1(x, y, |x-\bar{x}|), \qquad |q(x, y)-q(x, \bar{y})| \leqslant \varepsilon_2(x, y, |y-\bar{y}|).$$

On prouve sans peine que W est un sous-ensemble convexe et compact de l'espace de Banach formé de tous les couples $\{p, q\}$ de fonctions définies et continues dans Δ avec la norme

$$\|\{p, q\}\| = \max_{\Delta}|p(x, y|+\max_{\Delta}|q(x, y)|.$$

Des calculs bien simples que nous omettons conduisent à la conclusion que la transformation fonctionnelle

$$(13) \quad F\cdot\{p, q\} = \Big\{\int_{g(x)}^{y} f\big(x, t, z(x,t), p(x, t), q(x, t)\big)\,dt+G\big(x, z(x, g(x)), q(x, g(x))\big),$$
$$\int_{h(y)}^{x} f\big(s, y, z(s, y), p(s, y), q(s, y)\big)\,ds+H\big(y, z(h(y), y), p(h(y), y)\big)\Big\},$$

où

$$(14) \qquad z(x, y) = \mathring{z}+\int_{\mathring{x}}^{x} p(s, \mathring{y})\,ds+\int_{\mathring{y}}^{y} q(x, t)\,dt,$$

est continue et que $F\cdot W = W$. Donc, en vertu du théorème bien connu de Schauder, il existe un couple invariant $\{p, q\} = F\cdot\{p, q\} \in W$. La fonction (14) correspondante jouit évidemment de toutes les propriétés demandées.

*) Cela veut dire que les variations absolues des fonctions f, G et H ne surpassent pas $\omega(\delta)$ lorsqu'une variable indépendante quelconque augmente de δ.

6. Remarques. La conclusion de notre théorème subsiste si l'on remplace l'hypothèse $|g(x)-g(\bar{x})| \leqslant a|x-\bar{x}|$ et $|h(y)-h(\bar{y})| \leqslant b|y-\bar{y}|$ par la suivante, plus faible: $|g(x)| \leqslant a|x|$ et $|h(y)| \leqslant b|y|$ et si l'on renforce la condition (1) en admettant que $\varphi(\delta) = L\delta$, $0 < L = \text{const.}$

Dans ce cas, la démonstration n'exige qu'une légère modification de la méthode exposée plus haut. La différence essentielle consiste dans la définition des fonctions

$$\varepsilon_1(x, y, \delta) \quad \text{et} \quad \varepsilon_2(x, y, \delta)$$

qui est ici plus simple, à savoir

$$\varepsilon_1(x, y, \delta) = \sqrt{A}\cdot D(\delta)\, e^{L\lambda(x,y)}, \qquad \varepsilon_2(x, y, \delta) = \sqrt{B}\cdot D(\delta)\, e^{L\mu(x,y)},$$

où

$$\tilde{\Omega}(\delta) = M\cdot d(\delta) + (2a+2\beta+1)\cdot\omega\big((1+K+M)\delta + K\cdot d(\delta)\big),$$

$$D(\delta) = \left(\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}}\right)\left\{\tilde{\Omega}(\delta) + \sqrt{AB}\cdot\tilde{\Omega}\big(d(\delta)\big) + \big(\sqrt{AB}\big)^2\cdot\tilde{\Omega}\big(d(d(\delta))\big) + \ldots\right\}$$

et $d(\delta)$ désigne le module de continuité des fonctions $g(x)$ et $h(y)$ que nous pouvons évidemment supposer borné; la convergence uniforme du second membre est ainsi assurée.

En comparaison avec le Théorème 2 ([3], p. 582) dû à Mlle Z. Szmydt, cette modification de notre théorème concerne un cas beaucoup plus spécial — une seule équation, certaines restrictions relatives aux fonctions $g(x)$ et $h(y)$ — mais, d'autre part, nous n'avons introduit aucune limitation pour les constantes a, β, M et L, et nous avons ainsi démontré que, dans ce cas spécial, il existe une solution non locale, sans faire intervenir le procédé du prolongement d'une solution locale.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.— Vol. VI, No. 5. 1958

THEORETICAL PHYSICS

The Coulomb Effect in β Decay

by

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Presented by L. INFELD on March 1, 1958

Introduction

Because of the high electric charge of the nucleus one must take into account, in the β decay theory, the Coulomb effect in all β transitions, K-capture, etc. According to traditional methods [1], the plane wave representing β electrons should be replaced by an exact solution of the Dirac equation in the Coulomb field of the nucleus [2]. The exact Coulomb wave function is given in the jlm representation, and the calculations with the ψ_{jlm} wave function are much more complicated than the calculations with plane waves, especially in the polarization effects, electron-neutrino correlations, radiative β decay and other problems now playing an important role in connection with the nonconservation of parity. Difficulties in all these phenomena arise because the ψ_{jlm} wave function represents a spherical wave, and a matrix element constructed with it does not correspond to a definite electron momentum $\vec{p}$ and spin σ. The desired result is to be obtained in an indirect way based, for example, on the density matrix technique similar to that used in γ decay theory [3].

In this note we give an exact Coulomb wave function for electrons. This wave function possessing the full utility of the plane wave solution simplifies the calculations and obviates the introduction of a very complex density matrix [4].

Theory

Besides the ψ_{jlm} solution of the Dirac equation in the Coulomb field there are known solutions $\Psi(\vec{r}, \vec{p}, \sigma)$ depending on the asymptotic momentum $\vec{p}$ and spin σ, but such solutions were given only for high-energy electrons and light nuclei. Following Cutkosky's [7] idea we can write the wave function $\Psi(\vec{r}, \vec{p}, \sigma)$ in the form

$$\Psi(\vec{r}, \vec{p}, \sigma) = A(\vec{r}, \vec{p})u(\vec{p}, \sigma), \tag{1}$$

where $u(\vec{r}, \sigma)$ is a plane wave spinor and $A(\vec{r}, \vec{p})$ is an operator whose most general form for positive energies is (App. 1):

$$(2) \qquad A(\vec{r}, \vec{p}) = F_1(\vec{r}, \vec{p}) + \beta F_2(\vec{r}, \vec{p}) + i\vec{\alpha}\vec{r}\,[F_3(\vec{r}, \vec{p}) + \beta F_4(\vec{r}, \vec{p})]\,.$$

Operator A is determined if the four scalar functions $F_i(r, p)$ are known. For this purpose we develop the unknown wave function $\Psi(\vec{r}, \vec{p}, \sigma)$ into the complete set of functions $\psi_{k\mu}$ (App. 2) [8]:

$$(3) \qquad \Psi(\vec{r}, \vec{p}, \sigma) = \sum_{k\mu} C_{k\mu}\psi_{k\mu}\,,$$

where $C_{k\mu}$ are certain coefficients which will be calculated later. The functions $\psi_{k\mu}$ can be written in a form in which the radial and angular parts are separated

$$(4) \qquad \psi_{k\mu} = R_k(r)\Omega_{k\mu}(\theta, \varphi)\,,$$

where the symbols introduced above have the following meaning:

$$(5) \qquad R_k(r) = \begin{pmatrix} -ig_k & 0 \\ 0 & f_k \end{pmatrix}; \qquad \Omega_{k\mu} = \begin{pmatrix} \chi_k^\mu \\ \chi_{\bar{k}}^\mu \end{pmatrix}.$$

If we assume for the time being that the potential is equal 0, then expression (3) is the plane wave expansion into the set of $\psi_{k\mu}^0$ functions, the latter being solutions of the free Dirac equation. We have in this case

$$(6) \qquad u(\vec{p}, \sigma)e^{i\vec{p}\vec{r}} = \sum_{k\mu} a_{k\mu}\psi_{k\mu}^0 = \sum_{k\mu} a_{k\mu}R_k^0\Omega_{k\mu}\,,$$

where the superscript 0 denotes that the diagonal matrix R_k given by (5) contains free radial solutions given by (A_2). We shall obtain the plane wave expansion coefficients $a_{k\mu}$ from (6). One finds

$$(7) \qquad a_{k\mu} = \int \psi_{k\mu}^{0\dagger} u(\vec{p}, \sigma)e^{i\vec{p}\vec{r}}\, d^3x\,.$$

Expanding $e^{i\vec{p}\vec{r}}$ into spherical harmonics in the usual way, we obtain

$$(8) \qquad a_{k\mu} = \sum_{lm} 4\pi i^l \overset{*}{Y_l^m}\left(\frac{\vec{p}}{p}\right) \int \psi_{k\mu}^{0\dagger} Y_l^m\left(\frac{\vec{r}}{r}\right) j_l(pr)\, d^3x\, u(\vec{p}, \sigma)\,.$$

The coefficients (8) depend on $\vec{p}$ and σ; we shall perform here the integration only for the case $\sigma = +\frac{1}{2}$ (the case $\sigma = -\frac{1}{2}$ is not required for further purpose because it leads to the same result). We have

$$
(9)\qquad a_{k\mu}(\vec{p},\sigma=\tfrac{1}{2})=\frac{2\pi^2}{p\sqrt{\pi E}}i^k\times\left\{\begin{array}{ll}
i\left[\frac{\sqrt{E+1}}{E-1}\left(p_z\sqrt{\frac{k-\mu+\frac{1}{2}}{2k+1}}\,\overset{*}{Y}{}_k^{\mu-\frac{1}{2}}+\right.\right. & \\
\qquad\left.+p_+\sqrt{\frac{k+\mu+\frac{1}{2}}{2k+1}}\,\overset{*}{Y}{}_k^{\mu+\frac{1}{2}}\right)+ & \\
\qquad\left.+\sqrt{E-1}\sqrt{\frac{k+\mu-\frac{1}{2}}{2k-1}}\,\overset{*}{Y}{}_{k-1}^{\mu-\frac{1}{2}}\right] & k\geqslant 1\\
\left[\frac{\sqrt{E+1}}{E-1}\left(p_z\sqrt{\frac{k-\mu+\frac{1}{2}}{2k+1}}\,\overset{*}{Y}{}_{-k-1}^{\mu-\frac{1}{2}}-\right.\right. & \\
\qquad\left.-p_+\sqrt{\frac{k+\mu+\frac{1}{2}}{2k+1}}\,\overset{*}{Y}{}_{-k-1}^{\mu+\frac{1}{2}}\right)+ & \\
\qquad\left.+\sqrt{E-1}\sqrt{\frac{k+\mu-\frac{1}{2}}{2k-1}}\,\overset{*}{Y}{}_{-k}^{\mu-\frac{1}{2}}\right] & k\leqslant -1,
\end{array}\right.
$$

where the spherical harmonics depend on $\frac{\vec{p}}{p}$; the energy E and momentum $\vec{p}$ are measured in units of mc² and mc, respectively. We also put $h=c=1$. We see from (4) and (5) that the existence of any central potential is manifested in the solution $\psi_{k\mu}$ only by a change of the radial functions g_k and f_k in the diagonal matrix R_k, while the spinor $\Omega_{k\mu}$ remains unaltered. We can now determine the wave function (3) which fulfills the required asymptotic condition. When the desired form of wave function (3) at infinity is a plane wave plus an ingoing spherical wave (4), then the arbitrary coefficients $C_{k\mu}$ in Eq. (3) are simply

$$C_{k\mu}=a_{k\mu}e^{-i\Delta_k},$$

where $a_{k\mu}$ are the same as in (8) and Δ_k is the Coulomb phase shift given by (8), (4)

$$
(10)\qquad \Delta_k=\tfrac{1}{2}\arg\frac{-k+i\frac{aZ}{p}}{\gamma+iE\frac{aZ}{p}}-\arg\Gamma\left(\gamma+iE\frac{aZ}{p}\right)+\frac{\pi}{2}[l(-k)-\gamma],
$$

where $\gamma=\sqrt{1-(az)^2}$.

Our solution of the Dirac equation can be written in explicit form:

$$
(11)\qquad \Psi(\vec{r},\vec{p},\sigma)=\sum_{k\mu}a_{k\mu}e^{-i\Delta_k}\begin{pmatrix}-ig_k\chi_k^\mu\\ f_k\chi_k^\mu\end{pmatrix},
$$

where g_k and f_k are the Coulomb radial functions.

We can now find the functions $F_i(\vec{r},\vec{p})$ in the expression (2) if we compare (1) and (11). We have

$$
(12)\qquad [F_1+\beta F_2+i\vec{\alpha}\vec{r}(F_3+\beta F_4)]u(\vec{p},\sigma)=\sum_{k\mu}a_{k\mu}e^{-i\Delta_k}\begin{pmatrix}-ig_k\chi_k^\mu\\ f_k\chi_k^\mu\end{pmatrix}.
$$

Putting $\sigma=\frac{1}{2}$, we rewrite (12) in the matrix form

$$(13)\quad \begin{bmatrix} (F_1+F_2)\dfrac{p_z}{E-1}+iz(F_3-F_4) \\ (F_1+F_2)\dfrac{p_+}{E-1}+ix_+(F_3-F_4) \\ i(F_3+F_4)\dfrac{p_z z}{E-1}+i(F_3+F_4)\dfrac{x_- p_+}{E-1}+(F_1-F_2) \\ i(F_3+F_4)\dfrac{p_z x_+}{E-1}-i(F_3+F_4)\dfrac{p_+ z}{E-1} \end{bmatrix}=$$

$$=C\begin{bmatrix} p_z\left(-\dfrac{1}{p}g_{-1}e^{-i\Delta_{-1}}-i\dfrac{3g_{-2}e^{-i\Delta_{-2}}}{p^2 r}\vec{p}\,\vec{r}+\ldots\right)+iz\left(\dfrac{g_1e^{-i\Delta_1}+g_{-2}e^{-i\Delta_{-2}}}{r}+\ldots\right) \\ p_+\left(-\dfrac{1}{p}g_{-1}e^{-i\Delta_{-1}}-i\dfrac{3g_{-2}e^{-i\Delta_{-2}}}{p^2 r}\vec{p}\,\vec{r}+\ldots\right)+ix_+\left(\dfrac{g_1e^{-i\Delta_1}+g_{-2}e^{-i\Delta_{-2}}}{r}+\ldots\right) \\ ip_z z\left(\dfrac{f_2e^{-i\Delta_2}-f_{-1}e^{-i\Delta_{-1}}}{r}+\ldots\right)+ix_-p_+\left(\dfrac{f_2e^{-i\Delta_2}-f_{-1}e^{-i\Delta_{-1}}}{r}+\ldots\right)+ \\ \qquad\qquad +\left(-f_1e^{-i\Delta_1}-i\dfrac{3f_2e^{-i\Delta_2}}{pr}\vec{p}\,\vec{r}-\ldots\right) \\ ip_z x_+\left(\dfrac{f_2e^{-i\Delta_2}-f_{-1}e^{-i\Delta_{-1}}}{r}+\ldots\right)-ip_+z\left(\dfrac{f_2e^{-i\Delta_2}-f_{-1}e^{-i\Delta_{-1}}}{r}+\ldots\right) \end{bmatrix},$$

where the following notation is employed:

$$x_\pm=x\pm iy;\qquad p_\pm=p_x\pm ip_y;\qquad C=\frac{1}{p}\sqrt{\frac{\pi E}{E-1}}.$$

From (13) we obtain four independent algebraic equations *) for $F_i(\vec{r},\vec{p})$, solutions of which are **)

$$(14)\quad \begin{aligned} \left.\begin{matrix}F_1\\F_2\end{matrix}\right\} &= -\frac{E^{1/2}}{2p}\left[\frac{g_{-1}e^{-i\Delta_{-1}}}{\sqrt{E+1}}\pm\frac{f_1e^{-i\Delta_1}}{\sqrt{E-1}}\right]-\frac{E^{1/2}}{2p}\left[\frac{g_{-2}e^{-i\Delta_{-2}}}{\sqrt{E+1}}\pm\frac{f_2e^{-i\Delta_2}}{\sqrt{E-1}}\right]\frac{3i\vec{p}\,\vec{r}}{pr}+\ldots \\ \left.\begin{matrix}F_3\\F_4\end{matrix}\right\} &= \frac{E^{1/2}}{2pr}\left[\frac{f_2e^{-i\Delta_2}-f_{-1}e^{-i\Delta_{-1}}}{\sqrt{E+1}}\pm\frac{g_1e^{-i\Delta_1}+g_{-2}e^{-i\Delta_{-2}}}{\sqrt{E-1}}\right]+\ldots \end{aligned}$$

The functions F_i are expansions into powers of $\vec{p}\,\vec{r}$ and although they are written with such a degree of accuracy that terms of $(\vec{p}\,\vec{r})^2$ and higher orders are neglected, the procedure given above can be continued. Formulae (14) can also be generalized for an arbitrary central potential if the radial functions g_k and f_k and the phase shifts Δ_k are known. Eqs. (1), (2) and (14) give the wave function $\Psi(\vec{r},\vec{p},\sigma)$ that can be used in various problems to which the development into powers of $\vec{r}\vec{p}$ is relevant.

*) We should note that, by putting $\sigma=-\frac{1}{2}$ in (12), we obtain the same result.

**) In expressions (14) the radial functions f_k and g_k are normalized for a single particle in a unit sphere, as is often done in β decay theory.

Application to the β decay theory

The adaptation of wave function (1) to the β decay theory is based simply on taking all the coefficients of the expansion into powers of $\vec{r}\vec{p}$ in (14) on the nuclear surface, which is equivalent to the usual procedure. Let us write the transition matrix element with the help of the standard Hamiltonian [7]:

$$M(\vec{p}, \vec{q}, \sigma, \varrho) = \sum_{i=SPTVA} (V^* O_i u)(\Psi_e O_i \varphi_\nu),$$

$$\Psi_e = \Psi_e(\vec{r}, \vec{p}, \sigma); \quad \varphi_\nu = \varphi_\nu(\vec{r}, \vec{q}, \varrho),$$

where $\vec{p}, \vec{q}$ are the momenta and σ, ϱ the spins of the electron and neutrino, respectively. In terms of Eq. (1) we obtain

$$(15) \qquad M(\vec{p}, \vec{q}, \sigma, \varrho) = u_e^\dagger(\vec{p}, \sigma) \sum_i \int A^\dagger(\vec{r}, \vec{p}) e^{-i\vec{q}\,\vec{r}} (V^* O_i u)\, d^3x\, O_i u_\nu(\vec{q}, \sigma) =$$
$$= u_e^\dagger(\vec{p}, \sigma) \Lambda u_\nu(\vec{q}, \varrho),$$

where the Dirac matrices occurring in A operate on the lepton part, and a plane wave is taken as the neutrino wave function φ_ν. The process described by matrix element (15) corresponds to the Feynman graph given in Fig. 1. The vertex operator Λ involves both β and Coulomb interactions. We should mention that the Coulomb effect is included in Λ exactly and Fig. 1 is equivalent to an infinite sum of Born approximations as shown in Fig. 2. The matrix element of the desired process

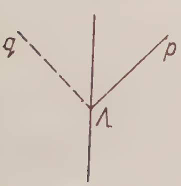

Fig. 1

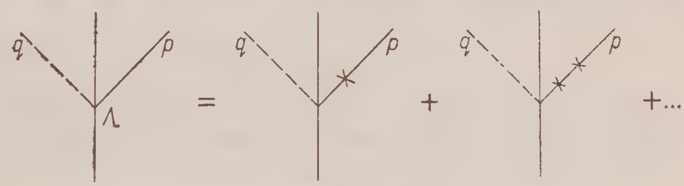

Fig. 2

depends on the momenta and spins of the outgoing particles, hence the spectra, angular correlations, and polarizations can be calculated with the help of the usual projection operators and trace techniques. Operator A occurring in the integrand of formula (15) contains terms necessary for allowed and first forbidden transitions. In the case of allowed transitions, a projection operator containing the Coulomb effect can be simply constructed. This operator is a generalization of the standard Casimir projection operator. As a matter of fact, the projection operator P is defined by

$$(16) \qquad P = \sum_{\text{Spins}} \Psi \Psi^+ = \sum A u u^+ A^+ = A P_c A^+,$$

where P_c is the Casimir operator $P_c = \dfrac{1}{2E}(E + \vec{\alpha}\vec{p} + \beta)$.

Inserting A from (2) and (14) and neglecting terms proportional to r we obtain, after some manipulations,

$$P = \frac{1}{2E}(E + \vec{\alpha}\vec{p} + \gamma\beta - i\lambda\beta\vec{\alpha}\vec{p})F(Z, E),$$

where

$$F(Z, E) = \frac{1}{2p^2}(g_{-1}^2 + f_1^2)|_{r=R}$$

is the usual Fermi function and

$$\gamma = \sqrt{1-(\alpha Z)^2}; \qquad \lambda = \frac{\alpha Z}{p}.$$

Jackson et al. [9], using a similar method, obtained precisely the same operator.

The method given above has been used to calculate the spectra and longitudinal polarizations of electrons for the allowed and first forbidden transitions. The results will not be presented here because they are identical with results previously obtained [1], [4], [10]. Some attempts to examine the internal Brehmstralung are in progress and will be published in a subsequent paper.

The author is much indebted to Dr. J. Werle for advice and helpful discussions during this work.

Appendix 1. An operator $A(\vec{r}, \vec{p})$ invariant to rotations is to be constructed from two vectors $\vec{r}$ and $\vec{p}$ and 16 Dirac matrices. Let us write all possible operators:

$$(A_1 1) \qquad \varrho_0 = I, \ \varrho_1, \ \varrho_2, \ \varrho_3, \ \varrho_0\vec{\sigma}\vec{r}, \ \varrho_1\vec{\sigma}\vec{r}, \ \varrho_2\vec{\sigma}\vec{r}, \ \varrho_3\vec{\sigma}\vec{r}.$$

The operators of the type $\varrho_i\vec{\sigma}\vec{p}$ are omitted because we can always reduce them to the ϱ_i operators in terms of the relation $\vec{\sigma}\vec{p}u(\vec{p}, \sigma) = \sigma p u(\vec{p}, \sigma)$. We divide $(A_1 1)$ into two sets:

(a) $\varrho_0, \ \varrho_3, \ \varrho_1\vec{\sigma}\vec{r}, \ \varrho_2\vec{\sigma}\vec{r}$;

(b) $\varrho_1, \ \varrho_2, \ \varrho_0\vec{\sigma}\vec{r}, \ \varrho_3\vec{\sigma}\vec{r}$.

Set (b) is to be expressed by set (a) multiplied by ϱ_1, but ϱ_1 is connected with ϱ_0 and ϱ_3 through the Dirac equation $(\varrho_1\sigma p + \varrho_3)u = \varrho_0 E u$. If the sign of E is kept constant, multiplication by ϱ_1 is equivalent to multiplication by ϱ_0 and ϱ_3, which again gives set (a). In conclusion, operator A is composed of operators from set (a) only.

Appendix 2. A solution [7] of the Dirac equation with an arbitrary central potential $V(r)$ can be written in bispinor form:

$$\Psi_{k\mu} = \begin{pmatrix} -ig_k\chi_k^\mu \\ f_k\chi_{-k}^\mu \end{pmatrix},$$

where $k = \pm 1, \pm 2, \ldots$ is the eigenvalue of the operator $K = -\varrho_3(\vec{\sigma}\vec{L} + 1)$, where $\vec{L} = -i\vec{r}\times\nabla$.

Quantum numbers j and l depend on k in accordance with the relations

$$j = |k| - \tfrac{1}{2},$$
$$l = |k| + \tfrac{1}{2}(S_k - 1),$$
$$S_k \equiv \operatorname{sign} k;$$

μ is the eigenvalue of J_z, the spinors χ_k^μ are expressed by spherical harmonics, the usual Pauli spinors $\chi_{\frac{1}{2}}^\mu$ being the eigenfunctions of $\vec{\sigma}^2$ and σ_z.

$$\chi_k^\mu = \sum_{\tau=-\frac{1}{2}}^{\frac{1}{2}} C_{\mu-\tau}^{l(k)\frac{1}{2}j(k)} \chi_{\frac{1}{2}}^\tau Y_{l(k)}^{\mu-\tau},$$

where $C_{\mu-\tau}^{l(k)\frac{1}{2}j(k)}$ are the Clebsch-Gordon coefficients. The radial functions $f_k(r)$ and $g_k(r)$ are solutions of the set of equations;

$$[E - V - 1]g_k = -\left(\frac{d}{dr} + \frac{1-k}{r}\right) f_k,$$

$$[E - V + 1]f_k = \left(\frac{d}{dr} + \frac{1+k}{r}\right) g_k.$$

For $V = 0$ we have the free solutions representing a standing wave normalized to the momentum interval

$$f_k^0 = \sqrt{\frac{E-1}{\pi E}}\, p^2 j_{l(k)}(pr),$$

$$g_k^0 = S_k \sqrt{\frac{E+1}{\pi E}}\, p^2 j_{l(-k)}(pr),$$

where the superscript 0 denotes the free solution and the $j_{l(k)}$ are spherical Bessel functions. The Coulomb radial functions are given in [2].

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 5. 1958

THEORETICAL PHYSICS

Two Theorems Concerning the Field Equations in the Spinor Space

by

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Presented by L. INFELD on March 10, 1958

1. Introduction

In a preceding paper [1] the connexion between the spinor and the vector space was studied in some detail. Here we want to establish the connexion between invariant field equations in both spaces.

According to formula (5.1) of paper [1], the connexion of the coordinates of the two spaces is

(1.1)
$$\begin{aligned}
X_1 &= \frac{\lambda}{2}(z_1 z_{\dot 2} + \overset{*}{z}_{\dot 1}\overset{*}{z}_2 + z_2 z_{\dot 1} + \overset{*}{z}_{\dot 2}\overset{*}{z}_1) + \frac{\varkappa}{2}(z_1\overset{*}{z}_2 + \overset{*}{z}_{\dot 1} z_{\dot 2} + z_2\overset{*}{z}_1 + \overset{*}{z}_{\dot 2} z_{\dot 1})\\
X_2 &= i\frac{\lambda}{2}(z_1 z_{\dot 2} + \overset{*}{z}_{\dot 1}\overset{*}{z}_2 - z_2 z_{\dot 1} - \overset{*}{z}_{\dot 2}\overset{*}{z}_1) + i\frac{\varkappa}{2}(z_1\overset{*}{z}_2 + \overset{*}{z}_{\dot 1} z_{\dot 2} - z_2\overset{*}{z}_1 - \overset{*}{z}_{\dot 2} z_{\dot 1})\\
X_3 &= \frac{\lambda}{2}(z_1 z_{\dot 1} + \overset{*}{z}_{\dot 1}\overset{*}{z}_1 - z_2 z_{\dot 2} - \overset{*}{z}_{\dot 2}\overset{*}{z}_2) + \frac{\varkappa}{2}(z_1\overset{*}{z}_1 + \overset{*}{z}_{\dot 1} z_{\dot 1} - z_2\overset{*}{z}_2 - \overset{*}{z}_{\dot 2} z_{\dot 2})\\
-X_0 &= \frac{\lambda}{2}(z_1 z_{\dot 1} + \overset{*}{z}_{\dot 1}\overset{*}{z}_1 + z_2 z_{\dot 2} + \overset{*}{z}_{\dot 2}\overset{*}{z}_2) + \frac{\varkappa}{2}(z_1\overset{*}{z}_1 + \overset{*}{z}_{\dot 1} z_{\dot 1} + z_2\overset{*}{z}_2 + \overset{*}{z}_{\dot 2} z_{\dot 2}).
\end{aligned}$$

Differentiating with respect to $z_\alpha, z_{\dot\alpha}, \overset{*}{z}_\alpha, \overset{*}{z}_{\dot\alpha}$ $(\alpha = 1, 2)$, and denoting

(1.2)
$$\tfrac{1}{2}(\lambda z_\alpha + \varkappa\overset{*}{z}_{\dot\alpha}) = a_\alpha, \qquad \tfrac{1}{2}(\lambda z_{\dot\alpha} + \varkappa\overset{*}{z}_\alpha) = a_{\dot\alpha},$$

we get

(1.3)
$$\begin{aligned}
&\frac{\partial X_1}{\partial z_1} = a_{\dot 2}, && \frac{\partial X_1}{\partial z_2} = a_{\dot 1}, && \frac{\partial X_1}{\partial z_{\dot 1}} = a_2, && \frac{\partial X_1}{\partial z_{\dot 2}} = a_1,\\
&\frac{\partial X_2}{\partial z_1} = ia_{\dot 2}, && \frac{\partial X_2}{\partial z_2} = -ia_{\dot 1}, && \frac{\partial X_2}{\partial z_{\dot 1}} = -ia_2, && \frac{\partial X_2}{\partial z_{\dot 2}} = ia_1,\\
&\frac{\partial X_3}{\partial z_1} = a_{\dot 1}, && \frac{\partial X_3}{\partial z_2} = -a_{\dot 2}, && \frac{\partial X_3}{\partial z_{\dot 1}} = a_1, && \frac{\partial X_3}{\partial z_{\dot 2}} = -a_2,\\
&\frac{\partial X_0}{\partial z_1} = -a_{\dot 1}, && \frac{\partial X_0}{\partial z_2} = -a_{\dot 2}, && \frac{\partial X_0}{\partial z_{\dot 1}} = -a_1, && \frac{\partial X_0}{\partial z_{\dot 2}} = -a_2,
\end{aligned}$$

and the corresponding sixteen complex conjugate equations.

For an arbitrary function depending on the spinor variables $z_a, z_{\dot a}, z^*_a, z^*_{\dot a}$, through the intermediary of the variables X_μ only, we have the relations

$$\frac{\partial}{\partial z_a} = \frac{\partial X_\mu}{\partial z_a}\frac{\partial}{\partial X_\mu}, \qquad \frac{\partial}{\partial z_{\dot a}} = \frac{\partial X_\mu}{\partial z_{\dot a}}\frac{\partial}{\partial X_\mu}, \qquad (a = 1, 2), \tag{1.4}$$

and the relations complex conjugate to (1.4).

Introducing (1.3) into (1.4) we obtain

$$\begin{aligned} \frac{\partial}{\partial z_1} &= a_{\dot 2}(\partial_1 + i\partial_2) + a_{\dot 1}(\partial_3 - \partial_0), \\ \frac{\partial}{\partial z_2} &= a_{\dot 1}(\partial_1 - i\partial_2) - a_{\dot 2}(\partial_3 + \partial_0), \end{aligned} \tag{1.5}$$

$$\begin{aligned} \frac{\partial}{\partial z_{\dot 1}} &= a_2(\partial_1 - i\partial_2) + a_1(\partial_3 - \partial_0), \\ \frac{\partial}{\partial z_{\dot 2}} &= a_1(\partial_1 + i\partial_2) - a_2(\partial_3 + \partial_0), \end{aligned} \tag{1.6}$$

where

$$\partial_\mu \equiv \frac{\partial}{\partial X_\mu}. \tag{1.7}$$

Let us assume now that the differential operators $\frac{\partial}{\partial z_a}, \frac{\partial}{\partial z_{\dot a}}, \frac{\partial}{\partial z^*_a}, \frac{\partial}{\partial z^*_{\dot a}}$ operate on an arbitrary function of $z_a, z_{\dot a}, z^*_a, z^*_{\dot a}$ which may depend on these variables, as well explicitly as through the intermediary of the variables X_μ. Differentiations with respect to the variables which occur explicitly are denoted by $\frac{\partial}{\partial z'_a}, \frac{\partial}{\partial z'_{\dot a}}$ etc.; differentiations with respect to the variables occurring implicitly by means of the X-s are denoted by $\frac{\partial}{\partial z''_a}, \frac{\partial}{\partial z''_{\dot a}}$, etc. Thus, for an arbitrary function we may write

$$\frac{\partial}{\partial z_a} = \frac{\partial}{\partial z'_a} + \frac{\partial}{\partial z''_a}, \qquad \frac{\partial}{\partial z_{\dot a}} = \frac{\partial}{\partial z'_{\dot a}} + \frac{\partial}{\partial z''_{\dot a}}. \tag{1.8}$$

According to formulae (1.5-6) we may in (1.8) replace the differentiations concerning the implicit variables by differentiations with respect to X_μ.

$$\begin{aligned} \frac{\partial}{\partial z_1} &= \frac{\partial}{\partial z'_1} + a_{\dot 2}(\partial_1 + i\partial_2) + a_{\dot 1}(\partial_3 - \partial_0), \\ \frac{\partial}{\partial z_2} &= \frac{\partial}{\partial z'_2} + a_{\dot 1}(\partial_1 - i\partial_2) - a_{\dot 2}(\partial_3 + \partial_0), \end{aligned} \tag{1.9}$$

$$\begin{aligned} \frac{\partial}{\partial z_{\dot 1}} &= \frac{\partial}{\partial z'_{\dot 1}} + a_2(\partial_1 - i\partial_2) + a_1(\partial_3 - \partial_0), \\ \frac{\partial}{\partial z_{\dot 2}} &= \frac{\partial}{\partial z'_{\dot 2}} + a_1(\partial_1 + i\partial_2) - a_2(\partial_3 + \partial_0). \end{aligned} \tag{1.10}$$

2. A theorem concerning first order equations

Let us introduce the spinor function

$$\Psi_a[X_\mu(z_a, ...), z_a, ...] \quad \Psi_{\dot{a}}[X_\mu(z_a, ...), z_a, ...]$$

of the spinor variables, and let us form the four invariant (with respect to transformations of the c_2-group) differential operations

$$\frac{\partial\psi_a}{\partial z_a}, \quad \frac{\partial\psi_a}{\partial z_a^*}, \quad \frac{\partial\psi_{\dot{a}}}{\partial z_{\dot{a}}}, \quad \frac{\partial\psi_{\dot{a}}}{\partial z_{\dot{a}}^*}, \tag{2.1}$$

where summation over the index a runs from 1 to 2. Using equations (1.9-10) and their complex conjugates we may write:

$$\begin{aligned} \frac{\partial\psi_a}{\partial z_a} - \frac{\partial\psi_a}{\partial z_a'} &= a_{\dot{1}}[(\partial_3-\partial_0)\psi_1+(\partial_1-i\partial_2)\psi_2]+a_{\dot{2}}[(\partial_1+i\partial_2)\psi_1-(\partial_3+\partial_0)\psi_2] \\ \frac{\partial\psi_a}{\partial z_a^*} - \frac{\partial\psi_a}{\partial z_a^{*\prime}} &= a_{\dot{1}}^*[(\partial_3-\partial_0)\psi_1+(\partial_1-i\partial_2)\psi_2]+a_{\dot{2}}^*[(\partial_1+i\partial_2)\psi_1-(\partial_3+\partial_0)\psi_2], \end{aligned} \tag{2.2}$$

$$\begin{aligned} \frac{\partial\psi_{\dot{a}}}{\partial z_{\dot{a}}} - \frac{\partial\psi_{\dot{a}}}{\partial z_{\dot{a}}'} &= a_1[(\partial_3-\partial_0)\psi_{\dot{1}}+(\partial_1+i\partial_2)\psi_{\dot{2}}]+a_2[(\partial_1-i\partial_2)\psi_{\dot{1}}-(\partial_3+\partial_0)\psi_{\dot{2}}] \\ \frac{\partial\psi_{\dot{a}}}{\partial z_{\dot{a}}^*} - \frac{\partial\psi_{\dot{a}}}{\partial z_{\dot{a}}^{*\prime}} &= a_1^*[(\partial_3-\partial_0)\psi_{\dot{1}}+(\partial_1+i\partial_2)\psi_{\dot{2}}]+a_2^*[(\partial_1-i\partial_2)\psi_{\dot{1}}-(\partial_3+\partial_0)\psi_{\dot{2}}]. \end{aligned} \tag{2.3}$$

Let us also write out explicitly the conventional Dirac equations for the spinor function $\psi_a(X_\mu)$, $\psi_{\dot{a}}(X_\mu)$.

$$\begin{aligned} (\partial_3-\partial_0)\psi_1+(\partial_1-i\partial_2)\psi_2 &= m\psi^{\dot{1}}, \\ (\partial_1+i\partial_2)\psi_1-(\partial_3+\partial_0)\psi_2 &= m\psi^{\dot{2}}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} (\partial_3-\partial_0)\psi_{\dot{1}}+(\partial_1+i\partial_2)\psi_{\dot{2}} &= m\psi^1, \\ (\partial_1-i\partial_2)\psi_{\dot{1}}-(\partial_3+\partial_0)\psi_{\dot{2}} &= m\psi^2. \end{aligned} \tag{2.5}$$

The comparison of (2.2-3) with (2.4-5) yields the following results:

1) Assume first that $\psi_a, \psi_{\dot{a}}$ is a solution of the Dirac equations (2.4-5) which does not depend explicitly on $z_a, z_{\dot{a}}, z_a^*, z_{\dot{a}}^*$. In this case the explicit derivatives $\frac{\partial\psi_a}{\partial z_a'}, \frac{\partial\psi_a}{\partial z_{\dot{a}}'}$, etc. vanish. Introducing (2.4-5) into (2.2-3) we obtain

$$\begin{aligned} \frac{\partial\psi_a}{\partial z_a} &= m a_{\dot{a}}\psi^{\dot{a}}, &\quad \frac{\partial\psi_a}{\partial z_a^*} &= m a_{\dot{a}}^*\psi^{\dot{a}}, \\ \frac{\partial\psi_{\dot{a}}}{\partial z_{\dot{a}}} &= m a_a\psi^a, &\quad \frac{\partial\psi_{\dot{a}}}{\partial z_{\dot{a}}^*} &= m a_a^*\psi^a. \end{aligned} \tag{2.6}$$

Any solution of the Dirac equation is also a solution of Eqs. (2.6).

2) Eqs. (2.6), however, are more general than the Dirac equations since they admit solutions which depend also explicitly on the spinor

variables. Let us assume that $\psi_a, \psi_{\dot a}$ is an arbitrary solution of (2.6) of this kind. Now, the explicit derivatives do not in general vanish. Introducing (2.6) into (2.2-3) we obtain

$$
\begin{aligned}
-\frac{\partial\psi_a}{\partial z'_a} &= a_{\dot 1}[(\partial_3-\partial_0)\psi_1+(\partial_1-i\partial_2)\psi_2-m\psi^{\dot 1}]+a_{\dot 2}[(\partial_1+i\partial_2)\psi_1-(\partial_3+\partial_0)\psi_2-\\
&\qquad -m\psi^{\dot 2}]\,,\\
-\frac{\partial\psi_a}{\partial z^{*\prime}_{\dot a}} &= a^*_{\dot 1}[(\partial_3-\partial_0)\psi_1+(\partial_1-i\partial_2)\psi_2-m\psi^{\dot 1}]+a^*_{\dot 2}[(\partial_1+i\partial_2)\psi_1-(\partial_3+\partial_0)\psi_2-\\
&\qquad -m\psi^{\dot 2}]\,,
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
-\frac{\partial\psi_{\dot a}}{\partial z'_{\dot a}} &= a_1[(\partial_3-\partial_0)\psi_{\dot 1}+(\partial_1+i\partial_2)\psi_{\dot 2}-m\psi^1]+a_2[(\partial_1-i\partial_2)\psi_{\dot 1}-(\partial_3+\partial_0)\psi_{\dot 2}-\\
&\qquad -m\psi^2]\,,\\
-\frac{\partial\psi_{\dot a}}{\partial z^{*\prime}_a} &= a^*_{\dot 1}[(\partial_3-\partial_0)\psi_{\dot 1}+(\partial_1+i\partial_2)\psi_{\dot 2}-m\psi^1]+a^*_{\dot 2}[(\partial_1-i\partial_2)\psi_{\dot 1}-(\partial_3+\partial_0)\psi_{\dot 2}-\\
&\qquad -m\psi^2]\,.
\end{aligned}
\tag{2.8}
$$

We notice first that if $\psi_a, \psi_{\dot a}$ is a solution of the Dirac equation, then the right hand sides of (2.7-8) vanish identically and the explicit dependence of $\psi_a, \psi_{\dot a}$ on $z_a, z_{\dot a}, z^*_a, z^*_{\dot a}$ is determined by the four independent equations

$$
\begin{aligned}
\frac{\partial\psi_a}{\partial z'_a} &= 0\,, \qquad \frac{\partial\psi_a}{\partial z^{*\prime}_{\dot a}} = 0\,,\\
\frac{\partial\psi_{\dot a}}{\partial z'_{\dot a}} &= 0\,, \qquad \frac{\partial\psi_{\dot a}}{\partial z^{*\prime}_a} = 0\,.
\end{aligned}
\tag{2.9}
$$

If $\psi_a, \psi_{\dot a}$ is not a solution of (2.4-5), and if the determinant $\Delta = a_{\dot 1}a^*_{\dot 2} - a_{\dot 2}a^*_{\dot 1}$ does not vanish identically,

$$
\Delta \equiv a_{\dot 1}a^*_{\dot 2} - a_{\dot 2}a^*_{\dot 1} \equiv \tfrac{1}{4}(\lambda^2-\varkappa^2)z^*_a z^{\dot a} \not\equiv 0\,,
\tag{2.10}
$$

we may solve Eqs. (2.7-8) with respect to the Dirac differential operations (2.4-5) and obtain

$$
\begin{aligned}
(\partial_3-\partial_0)\psi_1+(\partial_1-i\partial_2)\psi_2-m\psi^{\dot 1} &= \Delta^{-1}\left(a_{\dot 2}\frac{\partial\psi_a}{\partial z^{*\prime}_{\dot a}} - a^*_{\dot 2}\frac{\partial\psi_a}{\partial z'_a}\right)\\
(\partial_1+i\partial_2)\psi_1-(\partial_3+\partial_0)\psi_2-m\psi^{\dot 2} &= -\Delta^{-1}\left(a_{\dot 1}\frac{\partial\psi_a}{\partial z^{*\prime}_{\dot a}} - a^*_{\dot 1}\frac{\partial\psi_a}{\partial z'_a}\right),
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
(\partial_3-\partial_0)\psi_{\dot 1}+(\partial_1+i\partial_2)\psi_{\dot 2}-m\psi^1 &= -\Delta^{*-1}\left(a_2\frac{\partial\psi_{\dot a}}{\partial z^{*\prime}_a} - a^*_2\frac{\partial\psi_{\dot a}}{\partial z'_{\dot a}}\right)\\
(\partial_1-i\partial_2)\psi_{\dot 1}-(\partial_3+\partial_0)\psi_{\dot 2}-m\psi^2 &= \Delta^{*-1}\left(a_1\frac{\partial\psi_{\dot a}}{\partial z^{*\prime}_a} - a^*_1\frac{\partial\psi_{\dot a}}{\partial z'_{\dot a}}\right).
\end{aligned}
\tag{2.12}
$$

The result shows that the general solutions of Eqs. (2.6) satisfy equations of the Dirac type with certain additional terms determined by the explicit dependence of $\psi_a, \psi_{\dot{a}}$ on $z_a, z_{\dot{a}}, z_a^*, z_{\dot{a}}^*$.

Let us consider now the case when condition (2.10) is not satisfied, i. e. when

$$\Delta \equiv 0 . \tag{2.13}$$

Identity (2.13) may be realized in the three following ways [1]:

$$a_a = \tfrac{1}{2}(\lambda z_a + \varkappa z_{\dot{a}}^*) \equiv 0 , \tag{2.14}$$

or

$$a_{\dot{a}} = \tfrac{1}{2}(\lambda z_{\dot{a}} + \varkappa z_a^*) \equiv 0 , \tag{2.15}$$

or

$$a_{\dot{a}} = \tfrac{1}{2}(\lambda z_{\dot{a}} + \varkappa z_a^*) \equiv \varrho a_a^* = \varrho \tfrac{1}{2}(\lambda z_{\dot{a}}^* + \varkappa z_a) . \tag{2.16}$$

It is easily seen that these identities are equivalent with the following identities for the spinor variables $z_a, z_{\dot{a}}$ or for the parameters λ and $\varkappa$:

$$z_a \equiv 0, \quad \text{or} \quad z_{\dot{a}} \equiv 0, \quad \text{or} \quad z_{\dot{a}} \equiv c z_a^* , \quad \text{or} \quad \varkappa = \pm\lambda , \tag{2.17}$$

with c an arbitrary complex number.

From [1] (Sec. 3) we know that each of the four equations (2.17) is an equation of the light cone $X_\mu^2 = 0$. Thus, for points of the light cone, equations (2.8) are in general inconsistent unless one of the following alternatives is satisfied:

1) Eqs. (2.9) are satisfied.
2) The left hand sides of (2.7-8) satisfy the following relations

$$a_2^* \frac{\partial \psi_a}{\partial z_a'} - a_{\dot{2}} \frac{\partial \psi_a}{\partial z_{\dot{a}}^{*\prime}} = 0 \quad \text{or} \quad a_1^* \frac{\partial \psi_a}{\partial z_a'} - a_{\dot{1}} \frac{\partial \psi_a}{\partial z_{\dot{a}}^{*\prime}} = 0 , \tag{2.18}$$

and

$$a_{\dot{2}}^* \frac{\partial \psi_{\dot{a}}}{\partial z_{\dot{a}}'} - a_2 \frac{\partial \psi_{\dot{a}}}{\partial z_a^{*\prime}} = 0 \quad \text{or} \quad a_{\dot{1}}^* \frac{\partial \psi_{\dot{a}}}{\partial z_{\dot{a}}'} - a_1 \frac{\partial \psi_{\dot{a}}}{\partial z_a^{*\prime}} = 0 . \tag{2.19}$$

3. A theorem concerning second order equations

From (9) and (10) and the corresponding complex conjugate equations we may express ∂_μ by means of $\dfrac{\partial}{\partial z_a''}, \dfrac{\partial}{\partial z_{\dot{a}}''}$, etc. everywhere outside

the light cone, i. e. when condition (2.10) is satisfied. The result is

$$
\begin{aligned}
\partial_1 + i\partial_2 &= \Delta^{-1}\left(a_{\dot{1}} \frac{\partial}{\partial z_{\dot{1}}^{*\prime\prime}} - a_1^* \frac{\partial}{\partial z_1^{\prime\prime}}\right), \\
\partial_1 - i\partial_2 &= \Delta^{-1}\left(a_2^* \frac{\partial}{\partial z_2^{\prime\prime}} - a_{\dot{2}} \frac{\partial}{\partial z_{\dot{2}}^{*\prime\prime}}\right), \\
\partial_3 - \partial_0 &= \Delta^{-1}\left(a_2^* \frac{\partial}{\partial z_1^{\prime\prime}} - a_{\dot{2}} \frac{\partial}{\partial z_{\dot{1}}^{*\prime\prime}}\right), \\
\partial_3 + \partial_0 &= \Delta^{-1}\left(a_1^* \frac{\partial}{\partial z_2^{\prime\prime}} - a_{\dot{1}} \frac{\partial}{\partial z_{\dot{2}}^{*\prime\prime}}\right),
\end{aligned}
\tag{3.1}
$$

and the four equations complex conjugate to (3.1).

Constructing now, by means of (3.1), d'Alembert's differential operator $\Box \equiv \partial_\mu \partial_\mu$ and using relations (1.8-10), we get the following relation between the second order derivatives:

$$
\begin{aligned}
&\tfrac{1}{2}|\Delta|^{-2}\left(a_\alpha^* a_\beta \frac{\partial^2}{\partial z_\alpha^* \partial z_\beta} + \overset{*}{a}_{\dot{\alpha}} a_{\dot{\beta}} \frac{\partial^2}{\partial \overset{*}{z}_{\dot{\alpha}} \partial z_{\dot{\beta}}} - a_\alpha^* \overset{*}{a}_{\dot{\beta}} \frac{\partial^2}{\partial z_{\dot{\alpha}} \partial z_{\dot{\beta}}} - a_\alpha a_{\dot{\beta}} \frac{\partial^2}{\partial \overset{*}{z}_{\dot{\alpha}} \partial \overset{*}{z}_\beta}\right) = \Box + \\
&+ \tfrac{1}{2}|\Delta|^{-2}\left(a_\alpha^* a_\beta \frac{\partial^2}{\partial z_\alpha^{*\prime} \partial z_\beta^{\prime}} + \overset{*}{a}_{\dot{\alpha}} a_{\dot{\beta}} \frac{\partial^2}{\partial z_{\dot{\alpha}}^{*\prime} \partial z_{\dot{\beta}}^{\prime}} - a_\alpha^* \overset{*}{a}_{\dot{\beta}} \frac{\partial^2}{\partial z_{\dot{\alpha}}^{\prime} \partial z_\beta^{\prime}} - a_\alpha a_{\dot{\beta}} \frac{\partial^2}{\partial z_{\dot{\alpha}}^{*\prime} \partial z_\beta^{*\prime}}\right) - \\
&- \tfrac{1}{2}\Delta^{-1}\left(a_\alpha^* \frac{\partial}{\partial z_\beta^{\prime}} - a_{\dot{\alpha}} \frac{\partial}{\partial z_{\dot{\beta}}^{*\prime}}\right)\partial_\beta^{\dot{\alpha}} - \tfrac{1}{2}\Delta^{*-1}\left(a_\alpha \frac{\partial}{\partial z_\beta^{*\prime}} - \overset{*}{a}_{\dot{\alpha}} \frac{\partial}{\partial z_{\dot{\beta}}^{\prime}}\right)\partial_{\dot{\beta}}^{\alpha},
\end{aligned}
\tag{3.2}
$$

where $\partial_{\alpha\dot{\beta}}$ is shorthand for

$$
\begin{aligned}
\partial_{1\dot{1}} &= \partial_3 + \partial_0 & \partial_{1\dot{2}} &= \partial_1 - i\partial_2 \\
\partial_{2\dot{1}} &= \partial_1 + i\partial_2 & \partial_{2\dot{2}} &= -\partial_3 + \partial_0 .
\end{aligned}
\tag{3.3}
$$

From (3.2) we see that if φ is a solution of the Klein-Gordon equation

$$
(\Box - m^2)\varphi = 0 ,
\tag{3.4}
$$

depending on the variables X_μ only, then it satisfies with respect to the spinor variables the second order differential equation

$$
\tfrac{1}{2}|\Delta|^{-2}\left(a_\alpha^* a_\beta \frac{\partial^2}{\partial z_\alpha^* \partial z_\beta} + \overset{*}{a}_{\dot{\alpha}} a_{\dot{\beta}} \frac{\partial^2}{\partial \overset{*}{z}_{\dot{\alpha}} \partial z_{\dot{\beta}}} - a_\alpha^* \overset{*}{a}_{\dot{\beta}} \frac{\partial^2}{\partial z_{\dot{\alpha}} \partial z_\beta} - a_\alpha a_{\dot{\beta}} \frac{\partial^2}{\partial \overset{*}{z}_{\dot{\alpha}} \partial z_\beta^*}\right)\varphi = m^2\varphi ,
\tag{3.5}
$$

and *vice versa.*

In the general case, when $\varphi[X_\mu(z_\alpha, \ldots), z_\alpha, \ldots]$ is a function which depends on $z_\alpha, z_{\dot{\alpha}}, z_\alpha^*, \overset{*}{z}_{\dot{\alpha}}$, either explicitly or by means of X_μ, the general relation (3.2) holds. Assuming that the general φ satisfies an equation of the type (3.5), we obtain from (3.2) the following modification

of the Klein-Gordon equation:

$$(3.6)\qquad (\Box - m^2)\varphi = \tfrac{1}{2}|\Delta|^{-2}\left(a_\alpha a_{\dot\beta}\frac{\partial^2}{\partial z^{*\prime}_{\dot\alpha}\partial z^{*\prime}_{\dot\beta}} + a^*_\alpha a^*_{\dot\beta}\frac{\partial^2}{\partial z'_{\dot\alpha}\partial z'_\beta} - a^*_\alpha a_\beta\frac{\partial^2}{\partial z^{*\prime}_\alpha \partial z'_\beta} - \right.$$
$$\left. - a^*_{\dot\alpha} a_{\dot\beta}\frac{\partial^2}{\partial z^{*\prime}_{\dot\alpha}\partial z'_{\dot\beta}}\right)\varphi + \tfrac{1}{2}\Delta^{-1}\left(a^*_\alpha\frac{\partial}{\partial z'_\beta} - a_{\dot\alpha}\frac{\partial}{\partial z^{*\prime}_{\dot\beta}}\right)\partial^{\dot\alpha}_\beta\varphi + \tfrac{1}{2}\Delta^{*-1}\left(a_\alpha\frac{\partial}{\partial z^{*\prime}_\beta} - a^*_{\dot\alpha}\frac{\partial}{\partial z'_{\dot\beta}}\right)\partial^\alpha_{\dot\beta}\varphi.$$

We have thus obtained the connexion between invariant first and second order differential operations in both spaces. It may be noted that the Klein-Gordon and the Dirac differential operators are the only invariant linear second and first order differential operators in Minkowski's space. They appear as certain combinations of invariant linear differential operators in the spinor space. Each of the operators of the combination, however, may have some meaning of its own. Thus, the spinor space offers a greater variety of differential equations than the vector space. Of particular interest are, of course, differential operations corresponding to movements which leave the co-ordinates X_μ unchanged [1]. We shall give elsewhere a detailed study of such operations.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VI, No. 5. 1958

THEORETICAL PHYSICS

Excited Levels of Nuclei with Mixed Configurations. I. Method of Calculation

by

Z. SZYMAŃSKI

Presented by L. INFELD on March 10, 1958

Introduction

For a theoretical explanation of nuclear level schemes nuclear models *) are generally used: For nuclei with one nucleon or one hole outside double closed shell (or subshell) core, independent particle model with $(j-j)$ coupling (see for ex. [1]) is a suitable approximation. Level schemes of nuclei with more nucleons in unfilled subshells have a more complex structure, which is connected with mutual interactions of many external nucleons. The problem is to calculate excited levels for such nuclei treating residual interactions between all nucleon pairs as perturbation. This problem was solved by Flowers et al. [2]-[5] in the case of light nuclei, when protons and neutrons fill identical subshells (i. e. $j_p = j_n$). However, for nuclei with $A \geqslant 30$ protons and neutrons fill different subshells and the important influence of interaction between external nucleons from different subshells (i. e. subshells with $j_p \neq j_n$) should be taken into account.

In this intermediate mass region, only excited levels of odd-odd nuclei were calculated. These calculations were based on the so-called "odd-group" model [6], which takes into account both odd groups of nucleons with seniority quantum number equal to 1.

This note is an attempt to calculate excited levels of nuclei in the intermediate mass region on the assumption that protons and neutrons are placed in different subshells. Residual interactions between all external nucleons are assumed. Here we get out beyond the "odd-group" model, since neither seniority equal to one nor odd-odd nuclei are assumed **).

---

*) Only spherical nuclei will be taken into account.

**) Such a model was already used to find binding energies of nuclei [7].

It is also well known [8] that many low lying excited states of spherical even-even nuclei with even spins (2; 0, 2, 4) are interpreted as vibrational states. Such states are due to the simultaneous excitation of many nucleons. The question arises, whether it is possible to explain these levels on the ground of the model introduced here (i. e. by assuming that residual interactions of many nucleons may be treated as a perturbation).

Method

The wave function of the nuclear state with total angular momentum J, its z — component M belonging to a configuration containing P protons in the subshell j_p and N neutrons in the subshell j_n outside closed shells can be written generally as:

$$\Psi_J^M = \sum_{\substack{J_P J_N \\ \beta_P \beta_N}} A_{\beta_P J_P \beta_N J_N} \Psi\left((j_p)^P \beta_{\beta_P J_P}, (j_n)^N \beta_N J_N : JM\right), \tag{1}$$

where functions $\Psi\left((j_p)^P \ldots JM\right)$ form a base of the representation assumed. They are given by the vector addition proton and neutron functions (with total angular momentum J_P and J_N, respectively):

$$\Psi((j_p)^P \beta_P J_P, (j_n)^N \beta_N J_N : yM) = \\ = \sum_{M_P M_N} C^{JM}_{J_P M_P J_N M_N} \cdot \Psi\left((j_p)^P \beta_P J_P M_P\right) \cdot \Psi\left((j_n)^N \beta_N J_N M_N\right). \tag{2}$$

$C^{JM}_{J_P M_P J_N M_N}$ denotes the vector addition coefficient. The function $\Psi\left((j_p)^P \beta_P J_P M_P\right)$ is a completely antisymmetrized wave function of P protons in the shell j_p given by Edmonds and Flowers [3]. β_P denotes a set of remaining quantum numbers which make up together with J_P and M_P a complete set of quantum numbers (in the notation of [3], $\beta_P \equiv [\lambda](\sigma)\alpha$). The function $\Psi\left((j_n)^N \beta_N J_N M_N\right)$ is an analogical wave function for neutrons.

We assume that interactions between all pairs of external nucleons are given by the operator $\sum_{i<j} V_{ij}$, where V_{ij} denotes the two-body residual interaction operator, which can be written quite generally in the form:

$$V_{ij} = -v(r_{ij})[w + mP^x_{ij} + bP^\sigma_{ij} - hP^\sigma P^x]. \tag{3}$$

These interactions leave degeneracy in respect to these quantum numbers that specify the assumed representation (in particular in respect to J_P, J_N and J). To find the level scheme, operator $\sum V_{ij}$ must be diagonalized. This can be done by using functions (2) as a set of basic

vectors. Then we get the secular equation:

$$(4)\qquad \det\Big|\Big\langle (j_p)^P\beta'_P J'_P, (j_p)^N\beta'_N J'_N : JM\Big|\sum V_{ij}\Big|(j_p)^P\beta_P J_P, (j_n)\beta_N J_N : JM\Big\rangle - \\ - E\delta_{J'_P J_P}\delta_{J'_N J_N}\delta_{\beta'_P\beta_P}\delta_{\beta'_N\beta_N}\Big| = 0 .$$

Owing to the scalar character of the operator $\sum V_{ij}$ only diagonal matrix elements in respect to J occur. Then the secular equation is reduced to a set of secular equations of a considerably lower degree.

Only diagonal matrix elements (in respect to J_P and J_N are present for the V_{pp} and V_{nn} operators (i. e. operator V_{ij} in the cases of two protons or two neutrons, respectively). Then nondiagonal contributions come into account only from the matrix elements of operators V_{np} (i. e. neutron — proton operators). These matrix elements can be expressed in the form:

$$(5)\qquad \Big\langle (j_p)^P\beta'_P J'_P, (j_n)^N\beta'_N J'_N : JM\Big|\sum V_{np}\Big|(j_p)^P\beta_P J_P, (j_n)^N\beta_N J_N : JM\Big\rangle =$$

$$= P\cdot N\cdot \sum_{\substack{\beta_{P-1}\beta_{N-1}J_{P-1}\\ J_{N-1}J_a\;J_b}} \big((j_p)^P\beta'_P J'_P\{\big|(j_p)^{P-1}\beta_{P-1}J_{P-1}, j_p J'_P\big)\cdot$$

$$\cdot\big((j_n)^N\beta'_N J'_N\{\big|(j_n)^{N-1}\beta_{N-1}J_{N-1}, j_n J'_N\big)\cdot\big((j_p)^{P-1}\beta_{P-1}J_{P-1}, j_p J_P\big|\}(j_p)^P\beta_P J_P\big)\cdot$$

$$\cdot\big((j_n)^{N-1}\beta_{N-1}J_{N-1}, j_n J_N\big|\}(j_n)^N\beta_N J_N\big)\cdot$$

$$\cdot[(2J'_P+1)(2J'_N+1)(2J_P+1)(2J_N+1)]^{1/2}(2J_a+1)(2J_b+1)\cdot$$

$$\cdot\begin{Bmatrix} j_p & j_n & J_a\\ J_{P-1} & J_{N-1} & J_b\\ J'_P & J'_N & J\end{Bmatrix}\cdot\begin{Bmatrix} j_p & j_n & J_a\\ J_{P-1} & J_{N-1} & J_b\\ J_P & J_N & J\end{Bmatrix}\cdot\langle j_p j_n, J_a|V_{np}|j_p j_n, J_a\rangle .$$

Expressions:

$$\big((j)^m\beta_m J_m\{\big|(j)^{m-1}\beta_{m-1}J_{m-1}, jJ_m\big)$$

occuring in (5) denote fractional parentage coefficients [3]. Table

$$\begin{Bmatrix} a & b & e\\ c & d & f\\ g & h & k\end{Bmatrix}$$

denotes Wigner (9j) symbol. These symbols are partly tabulated [9]. They can also be computed from

$$(6)\qquad \begin{Bmatrix} a & b & e\\ c & d & f\\ g & h & k\end{Bmatrix} = \sum_\lambda (2\lambda+1)W(gkbd; h\lambda)\cdot W(ekcd, f\lambda)\cdot W(gcbe; a\lambda),$$

where symbols $W(abcd; ef)$ denote Racah coefficients.

The matrix elements

$$\langle j_p j_n J_a | V_{np} | j_p j_n J_a \rangle$$

may be calculated by standard methods in the form of linear combinations:

$$\langle j_p j_n J_a | V_{np} | j_p j_n J_a \rangle = \sum_k f_k F^{(k)}, \tag{7}$$

where $F^{(k)}$ are the Slater integrals. They can be computed by Thieberger's method [10] for various types of potentials $v(r_{ij})$ occurring in formula (3). Calculating matrix elements of operator V_{np} from (5) and matrix elements V_{pp}, V_{nn} – by the methods of work [3] we obtain the matrix element occurring in (4). Solutions of secular Eq. (4) give the energy levels and allow to find coefficients $A_{\beta_P J_P \cdot \beta_N J_N}$ which determine the true wave function Ψ_J^M of the nuclear state in the form of a linear combination (2) of basic functions. Knowledge of coefficients $A_{\beta_P J_P \beta_N J_N}$ is necessary for the calculation of transition probabilities.

Discussion

The method presented here enables to calculate nuclear excited level schemes. We can compare them with schemes obtainable from a simple addition of two corresponding Edmonds-Flowers [4] configurations $(j_p)^P \beta_P J_P$ and $(j_n)^N \beta_N J_N$ for protons and neutrons, respectively *). In such SM level scheme, individual levels are characterized by the numbers $(J_P J_N)$. They are degenerated in respect to the total angular momentum J, which can be obtained by vector addition J_P and J_N. The difference between SM level scheme and the scheme resulting from the method proposed here consist in the presence of matrix elements (5). They lead to configuration mixing. Nondiagonal contributions from the operator V_{np} are especially important.

Writing formula (5) in the form

$$\left\langle (j_p)^P \beta'_P J'_P, (j_n)^N \beta'_N J'_N : JM \left| \sum V_{np} \right| (j_p)^P \beta_P J_P, (j_n)^N \beta_N J_N : JM \right\rangle = \tag{8}$$

$$= P \cdot N \cdot \sum_{J_a} B(\beta'_P J'_P \beta'_N J'_N, \beta_P J_P \beta_N J_N, J_a J) \langle j_p j_n J_a | V_{np} | j_p j_n J_a \rangle ,$$

where quantity $B(\ldots)$ is defined by (5), we get:

$$\sum_{J_a} B(\beta'_P J'_P \beta'_N J'_N, \beta_P J_P \beta_N J_N, J_a J) = \delta_{\beta'_P \beta_P} \cdot \delta_{\beta'_N \beta_N} \cdot \delta_{J'_P J_P} \cdot \delta_{J'_N J_N} . \tag{9}$$

Consequently, the magnitude of nondiagonal matrix elements (in comparison with the diagonal ones) depends on the rate of change of

*) Such a level scheme will be called SM-scheme (shell model scheme).

the expression $\langle j_p j_n J_a | V_{np} | j_p j_n J_a \rangle$ with J_a. When the dependence of $\langle j_p j_n J_a | V_{np} | j_p j_n J_a \rangle$ on J_a is weak (i. e. if $\langle j_p j_n J_a | V_{np} | j_p j_n J_a \rangle \approx$ const.), the nondiagonal matrix elements vanish approximately. Then the influence of configuration interaction leads only to a downward shift of levels (i. e. in the direction of larger binding energy). This occurs, for example, in the case of Wigner forces ($m = b = h = 0$ in (3)) with range of considerable length.

Indeed, we have then

$$f_0 = 1$$

$$f_k F^{(k)} \ll 1 \qquad \text{for} \qquad k \neq 0 .$$

In this case excited levels are henceforth classified by seniority quantum numbers of protons and neutrons. Even for short range spin dependent forces nondiagonal matrix elements are small with respect to diagonal ones. However, they play an important role in the modification of the SM scheme. The nuclear state is a mixture of states of type (2) (i. e. $\beta_P, \beta_N, J_P, J_N$ are not good quantum numbers in principle). The degeneracy with respect to J is then removed.

When the range of forces is not too long it is found [4] that the first excited level of even-even nuclei has spin $2+$. Consequently, there are two close low lying $2+$ levels in the SM scheme (from protons and neutrons respectively). When configuration interaction of type (5) is put on, a repulsion of levels results (which was predicted by De Shalit and Goldhaber [11]). This leads to the following order of low lying levels: $0+$, $2+$, $2+$. It is also possible that one of the higher levels (with spin $0+$ or $4+$) will be shifted downwards below the second $2+$ level, as a result of the repulsion of levels. The orders of levels $0+$, $2+$, $4+$ or $0+$, $2+$, $0+$ are also possible. The three possibilities cited above correspond to the vibration level schemes structure in the spherical nuclei. However, any conclusions about the interpretation level schemes obtained here as the vibration schemes will be possible just now after estimation of electromagnetic transition probabilities. It will be a subject of further considerations.

It results from the above considerations that the influence of the configuration interaction play an important role only in cases, when nondiagonal elements (5) cannot be neglected in the secular equation (4). Such situation actually occurs when the corresponding Slater integrals $F^{(k)}(n_p l_p, n_n l_n)$ cannot be neglected, i. e. when a relative considerable overlapping of radial wave functions must occur. Unfortunately, in such cases one-particle shell model levels often lie close together. Consequently, j_p and j_n are not good quantum numbers (a mixing of one particle states occurs) and the perturbation method may be invalid. It will also be the subject of further considerations.

In the following note a numerical example for 4 external nucleons ($j_p = \frac{3}{2}, j_n = \frac{5}{2}$) will be presented.

The author expresses his sincere thanks to Professor L. Sliv for his kind hospitality at the Leningrad Institute of Physics and encouragement in the course of this work. He is also indebted to Dr. B. Birbrayer for many helpful discussions.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. —Vol. VI, No. 5. 1958

GEOPHYSICS

On a Cracovian Method of Determining the Regional Gravity Field

by

Z. FAJKLEWICZ

Presented by J. SAMSONOWICZ on March 12, 1958

The idea of applying mean square methods to the determination of the character of the regional gravity field originated in a paper by W. B. Agocs [1], and was subsequently developed by S. M. Simpson [4], who applied higher order polynomials in terms of least square counting. Orthogonal polynomials are used for the same purpose in a paper by C. H. G. Oldham and D. B. Sutherland [3]. Both methods [3], [4] necessitate high-speed electronic computers. It is the aim of the present paper to give an application of the least square method in Cracovian form to determine the character of the regional field. In this case, the Cracovian calculus yields the final result rapidly, there being no need of recurring to electronic computers.

In approximating the regional field function (Δg_R) the author uses second order polynomials only, since the application of higher order polynomials for this purpose would involve the possibility of an untrue mapping of the function Δg_R. This is a consequence of the fact that, the higher the order of the polynomial, the better it is suited to map the distribution of the gravity anomalies observed together with the local extrema, which is a disadvantage in the treatment of the present problem.

The regional gravity field may be written as second order polynomial in the following Cracovian form [2]:

$$\Delta g_R = \begin{Bmatrix} y^0 \\ y^1 \\ y^2 \end{Bmatrix} \begin{Bmatrix} a_{00} & a_{10} & a_{20} \\ a_{01} & a_{11} & 0 \\ a_{02} & 0 & 0 \end{Bmatrix} \begin{Bmatrix} x^0 \\ x^1 \\ x^2 \end{Bmatrix}, \tag{1}$$

wherein x, y are the co-ordinates of the observational values of the gravity field anomaly, and $a_{00}, a_{10}, \dots, a_{02}$ are its coefficients to be calculated. The polynomial (1) will yield the best approximation of the di-

stribution of the observational gravity field values (Δg) if the difference

$$\Delta g - \Delta g_R = \Delta g_L, \tag{2}$$

wherein Δg_L denotes the residual gravity, fulfills the conditions:

$$\sum_1^n \Delta g_{Ln} = \text{minimum} \tag{3}$$

which may be put in the form of normal equations; in Cracovian form, the latter may be written as follows:

$$\begin{Bmatrix} a_{20} \\ a_{02} \\ a_{11} \\ a_{10} \\ a_{01} \\ a_{00} \end{Bmatrix} \tau \begin{Bmatrix} x^4 & x^2y^2 & x^3y & x^3 & x^2y & x^2 \\ x^2y^2 & y^4 & xy^3 & xy^2 & y^3 & y^2 \\ x^3y & xy^3 & x^2y^2 & x^2y & xy^2 & xy \\ x^3 & xy^2 & x^2y & x^2 & xy & x \\ x^2y & y^3 & xy^2 & xy & y^2 & y \\ x^2 & y^2 & xy & x & y & x^0y^0 \end{Bmatrix} = \begin{Bmatrix} Gx^2 \\ Gy^2 \\ Gxy \\ Gx \\ Gy \\ G \end{Bmatrix} \tag{4}$$

This is a set of equations for the unknowns $a_{00}, a_{01} \dots a_{02}$, wherein the $x^i y^j$ are elements of the Cracovian of coefficients W and $G, Gx \dots$ – those of the Cracovian of free terms L, with G denoting the $\sum \Delta g$ upon the grid under consideration, and $Gx, Gy, Gxy \dots$ – the respective products of the co-ordinates and observational values of the gravity anomaly Δg.

According to T. Kochmański [2], system (4) may be solved by computing the inverse of the Cracovian of coefficients W. If the inverse is known, the co-efficients $a_{00}, a_{10}, \dots a_{02}$ are obtained directly by the following Cracovian operation:

$$a = L \cdot W^{-1}, \tag{5}$$

which is effected very rapidly.

The use of the implicit solution, which assumes knowledge of W^{-1} for calculating the unknowns presents the advantage that the inverse together with the squares grid for which it has been computed may be used in isolating the local gravity anomalies from the regional gravity field.

In order to calculate $a_{00}, a_{10}, \dots a_{02}$, it is necessary to have, except W^{-1}, the Cracovian of free terms L. Its elements $Gx^2, Gy^2 \dots$ may be rapidly computed using formulae which will be given presently.

Assuming the measurements of the gravity field anomaly to be interpolated in the corners of squares of sides $x = y = 1$, forming a grid over a rectangle of sides $P(y) \times Q(x)$, the formulae yielding the elements

of type $\sum G\alpha^n$ of the Cracovian L, in the co-ordinate system of Fig. 1, are of the form:

$$(6) \qquad \sum_1^Q Gx^p = x_1^p \sum g_1 + x_2^p \sum g_2 = \\ = x_3^p \sum g_3 + + + Q^p \sum g_Q$$

$$(7) \qquad \sum_1^P Gy^q = y_1^q \sum g_I + y_2^q \sum g_{II} = \\ = y_3^q \sum g_{III} + + + P^q \sum g_P .$$

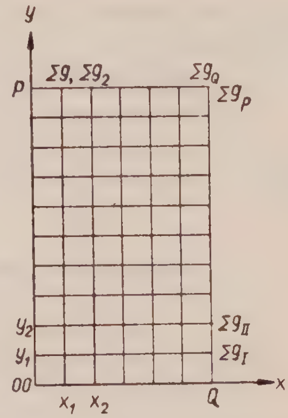

Fig. 1. Auxiliary graph for determining the parameters in formulae (6)–(9). The symbols $\sum g_1, \sum g_2, \ldots, \sum g_Q$ denote the sums of Δg in columns $x_1, x_2, \ldots, x_Q$. For $y_1, y_2, \ldots, y_P$ the sums are denoted by $\sum g_I, \sum g_{II}, \ldots, \sum g_p$.

In Eqs. (6) and (7) the notation is that of Fig. 1. For simplicity, the summation signs preceding the Cracovian elements in L and W have been omitted in (4). The following relations serve for computing the elements of type $\sum \alpha^n$ in the Cracovian of coefficients W:

$$(8) \quad \sum x^p = (P+1)\sum_{p/1}^{Q} x^p; \qquad \sum y^q = (Q+1)\sum_{q/1}^{P} y^q.$$

Those of type $\sum \alpha^n \beta^n$ are easily computed from:

$$(9) \qquad \sum x^p y^p = \sum_{p/1}^{Q} x^q \sum_{q/1}^{P} y^q .$$

When the elements of the Cracovian W have been computed from (8)-(9) for a squares grid covering a rectangle $P = 20 \times Q = 10$, the Cracovian may be written as follows:

	535 993	1 104 950	635 250	63 525	80 850	8 085
	1 104 950	7 949 326	2 425 500	175 850	485 100	31 570
(10)	635 250	2 425 500	1 104 950	80 850	157 850	11 550
	63 525	157 850	80 850	8 085	11 550	1 155
	80 850	485 100	157 850	11 550	31 570	2 310
	8 085	31 570	11 550	1 155	2 310	231

and its inverse in the form of the following table:

(11) TABLE

I	II	III	IV	V	VI	Control s.
43.444 73	+0.000 14	+ 0.000 22	—43.445 09	— 0.000 48	+ 6.517 06	+6.516 63
+ 0.000 14	+4.052 63	+ 0.000 25	— 0.000 38	— 8.105 41	+ 2.566 83	—1.485 94
+ 0.000 22	+0.000 25	+11.806 64	—11.806 89	— 5.903 85	+ 5.903 56	—0.000 08
—43.445 09	—0.000 38	—11.806 89	+59.581 15	+ 5.904 35	—14.585 16	—4.352 08
— 0.000 48	—8.105 41	— 5.903 85	+ 5.904 35	+20.343 42	— 9.266 11	+2.971 91
+ 6.517 06	+2.566 83	+ 5.903 56	—14.585 16	— 9.266 11	+ 8.250 85	—0.612 98

The last column gives the control sum of the values of the elements of the inverse.

When using the inverse, the following points should be kept in mind:

1. before use, the magnitudes

Gx^2, Gxy, Gy^2 should be multiplied by 10^{-6},
$Gx,\ Gy$ „ „ „ 10^{-5},
G „ „ „ 10^{-4};

2. computation yields the provisional unknowns

$$N_{20}, N_{02}, N_{11}, N_{10}, N_{01}, N_{00}$$

related to the final ones as follows:

$$\begin{aligned} N_{20} &= a_{20} & 10N_{10} &= a_{10}, \\ N_{02} &= a_{02} & 10N_{01} &= a_{01}, \\ N_{11} &= a_{11} & 100N_{00} &= a_{00}; \end{aligned}$$

3. on concluding the computation, the results are submitted to control by applying the following formula:

$$\varrho \cdot L = \sum_{n/1}^{6} N_n \tag{12}$$

wherein ϱ is the control sum in the Cracovian W^{-1},

L — the Cracovian of free terms,

$\sum N_n$ — the sum of provisional unknowns.

In order to provide an example, the method was applied to the region of the gravity anomaly of the environments of Poznań (Fig. 2), assuming a squares grid of 1,5 km. over a rectangle of 30×15 km.; using the Cracovian (11) and formulae (6) and (7), the following relations yielding the regional gravity was rapidly obtained:

$$\Delta g_R = +0{,}028x^2 - 0{,}029y^2 - 0{,}082xy + +0{,}576x + 0{,}483y + 4.208. \tag{13}$$

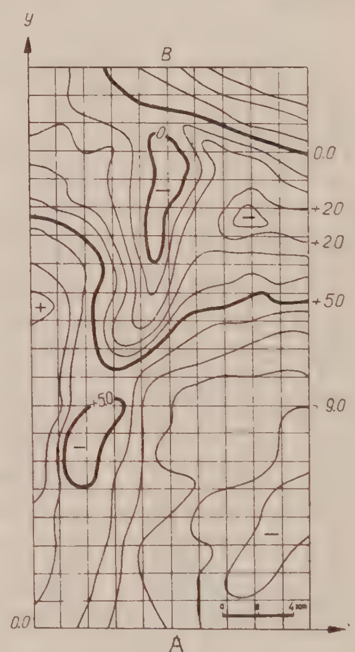

Fig. 2. Field of observed gravity anomalies of the environments of Poznań, according to Bouguer, prepared by W. Duda (PPG). Property of the Geological Institute, Warsaw

The regional gravity field underlying the gravity anomaly of the environments of Poznań calculated from Eq. (13) is given in Fig. 3. The map of the residual gravity, the values of which are obtained from Eq. (2), is shown in Fig. 4. In order to form a clear idea of the relation of the field Δg to the calculated regional gravity field and to obtain

a representation of its curvature, the profile AB (Fig. 5) was prepared. The latter is to serve no further purpose, since, in order to give a quantitative interpretation, it would be necessary to obtain the profiles through both negative extremes of the local anomalies in Fig. 4.

The principal step distinguishing the employement of the present method from that of the American methods adapted to the use of elec-

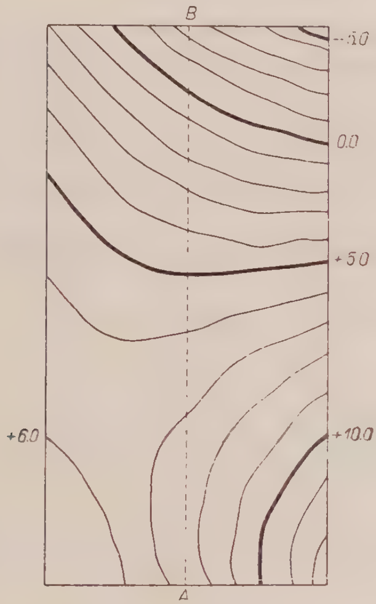

Fig. 3. Regional gravity field of Poznań anomaly in the form of a system of second order polynomial curves obtained by the Cracovian method

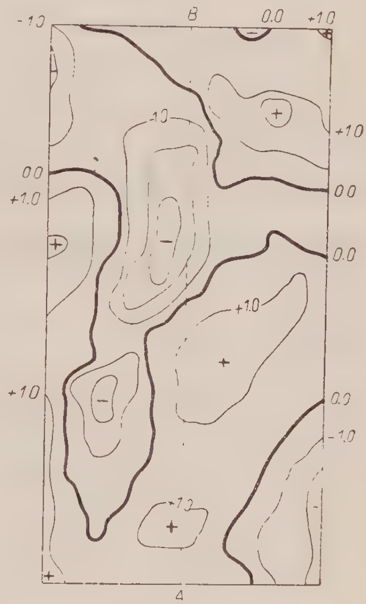

Fig. 4. Local anomalies of the environments of Poznań related to the regional gravity field shown in Fig. 3

tronic computers consists in the calculation of the inverse of the Cracovian of coefficients. Assuming, however, the inverses of the Cracovians to have been calculated in advance for the grids best suited to the pur-

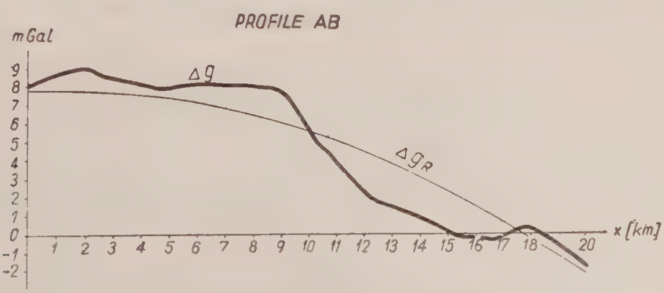

Fig. 5. Curve of the observed gravity anomaly as compared to the regional field along north-south profile

poses of gravimetry, the entire procedure of determining the shape of the regional field, a task solved by the Americans with the use of electronic computers, becomes a matter of about 20 minutes.

The present paper brings the inverse of the Cracovian of coefficients computed for a squares grid covering a rectangle $P = 20$, $Q = 10$. A catalogue tabulating the inverse of the Cracovians of coefficients for the following lattices:

$$\begin{matrix} Q=10 & Q=10 & Q=10 & Q=15 & Q=15 & Q=20 \\ P=10 & P=15 & P=20 & P=15 & P=20 & P=20 \end{matrix}$$

is in preparation. The tables will make it possible simultaneously to obtain the interpretation of gravimetric data over regions of various magnitudes up to 1800 km.^{2} ($P = 20$, $Q = 20$ with unit sides of 2 km.).

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VI 1958 ВЫПУСК 5

С. ГЛИДМАН, НЕСКОЛЬКО ТЕОРЕМ ОБ ОПРЕДЕЛИТЕЛЯХ стр. 275–280

Через $\bar{a}\cdot\bar{b} = (\bar{a}\bar{b})$ обозначим скалярное произведение двух векторов n-мерного векторного пространства X_n над полем действительных чисел. Через $\bar{w} = \bar{a}\times\bar{b}$ обозначим внешнее произведение этих векторов $\left(w_i = a_k b_l - a_l b_k\right.$ при $k<l$ и $\left. i = 1, 2, \ldots, \frac{n(n-1)}{2}\right)$.

Доказывается следующее равенство

$$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = (\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d}) - (\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c})\,. \tag{1}$$

Пусть $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n \in X_n$. Определяются следующие функции:

$$\bar{a}_{11,s} = \bar{a}_1\times\bar{a}_{s+1}\,, \qquad \bar{a}_{k,s} = \bar{a}_{k-1,1}\times\bar{a}_{k-1,\,s+1}\,. \tag{2}$$

Методом индукции доказывается, что

$$\bar{a}_{k,s} = \bar{a}_{k,s}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k, \bar{a}_{k,s}) \tag{3}$$

$$\bar{a}_{k,t} = \bar{a}_{k,s}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k, \bar{a}_{k,t}) \tag{4}$$

$$\bar{a}_{k,s}(\bar{a}_{l,1}, \bar{a}_{l,2}, \ldots, \bar{a}_{l,k}, \bar{a}_{l,k+s}) = \bar{a}_{k+l,s}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{k+l}, \bar{a}_{k+l+s})\,. \tag{5}$$

Используя равенства (1)-(5), доказывается, что произведение двух определителей $A = |a_{ik}|_n$ и $B = |b_{ik}|_n$, при $(\bar{a}_{k1}\beta_{k1}) \neq 0$ имеет вид:

$$AB = \frac{(\bar{a}_{n-1,1}\bar{\beta}_{n-1,1})}{(\bar{a}_1\bar{b}_1)^{n-2}(\bar{a}_{11}\bar{\beta}_{11})^{n-3}\ldots(\bar{a}_{n-3,1}\bar{\beta}_{n-3,1})}\,, \tag{6}$$

где функции $\beta_{k,s}$ определены аналогичным образом, как и функции $\alpha_{k,s}$.

Доказывается следующее обобщение известного неравенства Адамара [3]:

$$G_n \leqslant \frac{\prod\limits_{\nu=1}^{n-k} G_{k,\nu}}{G_k^{n-k-1}}\,, \tag{7}$$

где G_k определитель Грама от векторов $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$, а $G_{k,\nu}$ определитель Грама от векторов $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k, \bar{a}_{k+\nu}$ $(\nu = 1, 2, \ldots, n-k)$. Очевидно, что $G_{k+1} = G_{k,1}$.

Кроме того, приводится геометрическая интерпретация полученных результатов и исследуется внутренняя связь между неравенством (7) и оценками определителя, полученными Фаге [5] и Вегнером [6].

К. КУРАТОВСКИЙ, РАСШИРЕНИЕ ПОНЯТИЯ РАЦИОНАЛЬНОЙ ФУНКЦИИ НА n-МЕРНОЕ ЕВКЛИДОВО ПРОСТРАНСТВО . . . . стр. 281–287

Рассматриваются непрерывные функции f, определенные на подмножестве пространства E^n такие, что $0 \neq f(x) \in E^n$. Когомотопическое умножение Борсука позволяет ввести рациональные функции „в смысле гомотопии". Указывается, что многие теоремы, доказанные до сих пор для E^2, обобщаются на E^n.

З. СЕМАДЕНИ, ЛОКАЛИЗАЦИОННАЯ ТЕОРЕМА ДЛЯ МУЛЬТИПЛИКАТИВНЫХ ЛИНЕЙНЫХ ФУНКЦИОНАЛОВ . . . . стр. 289–292

Пусть Y некоторый класс ограниченных функций, определенных в некотором абсолютно регулярном пространстве T, замкнутом относительно сложения и существенной верхней грани (essential supremum) конечного числа функций, а также замкнутом относительно равномерной сходимости. Автор рассматривает фактор-пространство Y/R, пренебрегая значениями функций из Y на множествах, принадлежащих к некоторому фиксированному σ-идеалу граничных множеств R и вводит следующую дефиницию: точку $t_0 \in T$ называем точкой локализации линейно-мультипликативного функционала ξ, если для произвольной окрестности U точки t_0 и для произвольных $x, y \in Y/R$ условие

$$x(t) = y(t) \quad \text{для} \quad t \in U \setminus A,$$

где A — некоторое множество R, влечет за собой $\xi(x) = \xi(y)$.

Автор приводит ряд теорем (без доказательств), из которых наиболее важны следующие: 1° *для всякого t имеется линейно-мультипликативный функционал локализированный в t*; 2° *если T компактно, то всякий линейно мультипликативный функционал имеет свою точку локализации.*

М. АЛЬТМАН, РАСПРОСТРАНЕНИЕ НА ЛОКАЛЬНО ВЫПУКЛЫЕ ПРОСТРАНСТВА ТЕОРЕМЫ БОРСУКА ОБ АНТИПОДАХ . . . . стр. 293–295

Классическая теорема Борсука об антиподах была доказана М. А. Красносельским для вполне непрерывных векторных полей на сфере пространства Банаха.

В настоящей заметке теорема об антиподах доказывается для случая вполне непрерывного перемещения выпуклой симметричной окрестности нуля в локально выпуклом линейном топологическом пространстве.

Этот результат содержит, как частный случай, вышеупомянутую теорему Красносельского.

Как следствие теоремы об антиподах получается теорема о существовании неподвижной точки для вполне непрерывного преобразования в локально выпуклом пространстве.

М. АЛЬТМАН, НЕПРЕРЫВНЫЕ ПРЕОБРАЗОВАНИЯ ОТКРЫТЫХ МНОЖЕСТВ В ЛОКАЛЬНО ВЫПУКЛЫХ ПРОСТРАНСТВАХ . . стр. 297–301

Настоящая заметка содержит обобщение известной теоремы Борсука об ϵ-отображениях конечномерного евклидова пространства. Одновременно получается обобщение результатов А. Гранаса, которые в свою очередь обобщают на пространство Банаха теорему Борсука.

Обобщение данное в этой заметке касается случая локально выпуклого линейного топологического пространства. Основной результат состоит в указании условий достаточных для того, чтобы вполне непрерывное перемещение области в локально выпуклом пространстве являлось областью в этом пространстве.

Попутно получается простое доказательство теоремы об инвариантности области в локально выпуклом пространстве.

Доказательства существенным образом опираются на теорему об антиподах в локально выпуклом пространстве, доказанную в предыдущей заметке.

Г. ШТЕЙНГАУЗ, О НАИБОЛЕЕ КОРОТКИХ ПУТЯХ НА ЗАМКНУТЫХ ПОВЕРХНОСТЯХ . . . . стр. 303–308

Заметка касается замкнутых поверхностей, т. е. гомеоморфных обычной сфере и регулярных, т. е. таких, что всякая область Жордана на них является взаимно-однозначной и непрерывным образом плоской области $u^2+v^2<1$ посредством функций (1) с непрерывными производными до третьего порядка включительно и имеющих положительную дифференциальную форму $EG-F^2$.

Как известно, всякие две точки P, Q, лежащие на такой поверхности, можно соединить на ней наиболее коротким путем из всех путей, соединяющих P с Q на этой поверхности. Длину этого пути назовем расстоянием между этими двумя точками. Благодаря такому определению, можно говорить о точке, наиболее отдаленной от данной точки P.

Предметом настоящей заметки было утверждение, что из данной точки P всегда ведут, по крайней мере, два разных наиболее коротких путей к наиболее отдаленной точке. В заметке дается подробное доказательство этого утверждения.

С. МРУВКА, ЗАМЕТКА О МУЛЬТИПЛИКАТИВНО-ЛИНЕЙНЫХ ФУНКЦИОНАЛАХ . . . . стр. 309–311

Пусть $\mathfrak{m}$ — кардинальное число. $\mathfrak{m}$ — аддитивной мерой для множества T будем называть функцию μ, определенную на 2^T, принимающую значения 0 и 1 и удовлетворяющую следующим условиям: 1^0 $\mu(T)=1$, $\mu(\varnothing)=0$; 2^0 $\mu(A)\leqslant\mu(B)$ для $A\subset B\subset T$; 3^0 если $\mu(A)=1$, то $\mu T\setminus A=0$; 4^0 $\mu(U\{A:A\in\mathfrak{A}\})=\sup\{\mu(A):A\in\mathfrak{A}\}$ для всякого $\mathfrak{A}\subset 2^T$, где $\overline{\overline{\mathfrak{A}}}\leqslant\mathfrak{m}$.

Через $\aleph(\mathfrak{m})$ обозначим наименьшее кардинальное число $\mathfrak{n}$, для которого на множестве T мощности $\mathfrak{n}$ существует $\mathfrak{m}$-аддитивная мера μ, такая, что $\mu(\{t\}) = 0$ при всяком $t \in T$.

Пусть $R_t (t \in T)$ — система абстрактных алгебр над полем K и R — декартовое произведение алгебр R_t. В работе доказано, что

1° Если $\overline{\overline{T}} < \aleph(\overline{\overline{K}})$, то всякий гомоморфизм f алгебры R в поле K имеет вид

$$(*) \qquad f(x) = f_{t_0}(x(t_0)); \qquad (x \in R),$$

где t_0 — фиксированный элемент T и f_{t_0} — фиксированный гомоморфизм алгебры R_{t_0}.

2° Если для некоторого кардинального числа $\mathfrak{p}$, $\overline{\overline{K}} < \aleph(\mathfrak{p})$ и $\overline{\overline{T}} \geqslant \aleph(\mathfrak{p})$, то существует система $R_t (t \in T)$, такая, что на R существуют гомоморфизмы, не имеющие вида (*).

Следствиями этих теорем являются все результаты, данные в работе [1].

Ч. БЕССАГА и А. ПЭЛЧИНСКИЙ, О ПОДПРОСТРАНСТВАХ ПРОСТРАНСТВА С БЕЗУСЛОВНЫМ БАЗИСОМ . . . . стр. 313–315

Пусть X — пространство Банаха с безусловным базисом. Пусть Y — подпространство X. При этих предположениях имеют место следующие теоремы:

Теорема 1. *Следующие условия равносильны:*

(11) *в пространстве Y всякое ограниченное множество содержит слабо сходящуюся последовательность (не непременно к элементу пространства)*

(12) *Y^* — слабо полное*

(13) *Y^* — не содержит подпространства изоморфного к пространству c_0*

(14) *Y — не содержит дополнительного подпространства изоморфного к пространству l*

(15) *если Y_1 сепарабельное подпространство пространства Y, то Y_1^* — сепарабельное*

(16) *Y не содержит подпространства изоморфного к пространству l.*

Теорема 2. *Пространство Y — слабо полное тогда и только тогда, если Y не содержит подпространства изоморфного к пространству c_0.*

Теорема 3. *Следующие условия равносильны:*

(31) *Y — рефлексивное*

(32) *всякое ограниченное множество в Y^* содержит слабо сходящуюся последовательность*

(33) *Y^* не содержит подпространства изоморфного к пространству l*

(34) *Y не содержит подпространства изоморфного к пространству l или к c_0.*

Б. ПИЛАТ, ЗАМЕТКА О ЭКСТРЕМУМАХ ФУНКЦИЙ, СОСТАВЛЕННЫХ ИНТЕГРАЛАМИ УРАВНЕНИЙ С ЧАСТНЫМИ ПРОИЗВОДНЫМИ ЭЛЛИПТИЧЕСКОГО ТИПА . , ... стр. 317–319

В работе доказывается следующая теорема:

Пусть $\Delta_u^* \equiv \sum_{i,k=1}^{n} a_{ik}(x_1, \ldots, x_n)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x_1, \ldots, x_n)\frac{\partial u}{\partial x_i}$, где коэффициенты $a_{ik}(x_1, \ldots, x_n)$, $b_i(x_1, \ldots, x_n)$ непрерывны и ограничены в области D, форма $\sum_{i,k=1}^{n} a_{ik}\lambda_i\lambda_k$ положительно определенная, а ее определитель в D больше, чем положительная постоянная. Если функции $g(x_1, \ldots, x_n)$, $h_\nu(x_1, \ldots, x_n)$, $\nu = 1, 2, \ldots, s$, принадлежат к классу $C^{(2)}$ в D и имеются в D неравенства $\operatorname{sign} h_\nu \cdot \Delta^* h_\nu \leqslant 0$, то функция

$$w(x_1, \ldots, x_n) = e^{g(x_1,\ldots,x_n)} h_1(x_1, \ldots, x_n) \ldots h_s(x_1, \ldots, x_n)$$

не принимает ни в какой точке P области D ни относительного положительного минимума w_1, ни относительного отрицательного максимума w_2, если она не сводится к постоянной во всякой области D', содержащей точку P и заключающейся в D, в которой выполняется неравенство $w \geqslant w_1$ или $w \leqslant w_2$. Если дополнительно имеется в D неравенство $\Delta^* g < 0$ или $\operatorname{sign} h_\nu \cdot \Delta^* h_\nu < 0$ для какого нибудь $\nu(\nu = 1, 2, \ldots, s)$, то $w(x_1, \ldots, x_n)$ не принимает в D ни относительного положительного минимума, ни относительного отрицательного максимума.

Затем, приводится пример, в котором доказывается, что в случае, если $\Delta^* = \Delta$ является оператором Лапласа, то сумма выражений типа $e^g h_1, \ldots, h_s$, обладающих таким же знаком, может не обладать свойством, присущим ее слагаемым.

А. БЕЛЕЦКИЙ и Я. КИСЫНСКИЙ, О ПРОБЛЕМЕ С. ШМИДТ, КАСАЮЩЕЙСЯ УРАВНЕНИЯ $\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$ . . . . стр. 321–325

Пусть Δ — прямоугольник: $|x| \leqslant \alpha$, $|y| \leqslant \beta$; D — область; $(x, y) \in \Delta$, z, p, q произвольные; $f(x, y, z, p, q)$ — функция, ограниченная и непрерывная в области D и удовлетворяющая там же условию $|f(x, y, z, p, q) - f(x, y, z, \bar{p}, \bar{q})| \leqslant \varphi(|p - \bar{p}| + |q - \bar{q}|)$, где $\varphi(\delta)$ — некоторая неубывающая и непрерывная функция для $\delta \geqslant 0$, равная нулю при $\delta = 0$ и притом такая что для $\delta > 0$ интеграл $\int_0^\delta du/\varphi(u)$ расходится. Предполагаем, кроме того, что функции $G(x, z, q)$ и $H(y, z, p)$ — ограничены и непрерывны и удовлетворяют неравенствам

$$|G(x, z, q) - G(x, z, \bar{q})| \leqslant A \cdot |q - \bar{q}|, \quad |H(y, z, p) - H(y, z, \bar{p})| \leqslant B \cdot |p - \bar{p}|,$$

где $A > 0$, $B > 0$ и $AB < 1$, для $|x| \leqslant \alpha$ и $|y| \leqslant \beta$.

Предполагаем далее, что функции $g(x)$ и $h(y)$, определенные и непрерывные для $|x| \leqslant \alpha$ и $|y| \leqslant \beta$, удовлетворяют условиям $|g(x)| \leqslant \min(\beta, a|x|)$, $|h(y)| \leqslant \min(\alpha, b|y|)$, $a > 0$, $b > 0$, $ab < 1$, числа же $\mathring{x}$, $\mathring{y}$ и $\mathring{z}$ — условию $(\mathring{x}, \mathring{y}) \in \Delta$.

Принимаем наконец, что справедливо одно из двух условий; или для $\delta \geqslant 0$ $\varphi(\delta) = L\delta$ и $0 < L < +\infty$, или же $|g(x) - g(\bar{x})| \leqslant a \cdot |x - \bar{x}|$ и $|h(y) - h(\bar{y})| \leqslant b \cdot |y - \bar{y}|$ для $|x| \leqslant \alpha$, $|\bar{x}| \leqslant \alpha$ и $|y| \leqslant \beta$, $|\bar{y}| \leqslant \beta$. При таких предположениях имеется функция $z(x, y)$ определенная и непрерывная вместе со своими производными $\partial z/\partial x$, $\partial z/\partial y$, $\partial^2 z/\partial x \partial y$ в Δ, удовлетворяющая там уравнению упомянутому в заглавии работы, а также граничным условиям $\partial z/\partial x = G(x, z, \partial z/\partial y)$ для $|x| \leqslant \alpha$, $y = g(x)$ и $\partial z/\partial y = H(y, z, \partial z/\partial x)$ для $|y| \leqslant \beta$, $x = h(y)$, причем $z(\mathring{x}, \mathring{y}) = \mathring{z}$. Этот результат не совпадает с результатами, полученными С. Шмидт ([2] и [3]), в связи с вышеуказанным уравнением, но в некотором смысле частично лишь их обобщает.

А. ДЕЛОФФ, КУЛОНЭФФЕКТ ПРИ РАСПАДЕ β . . . . стр. 327—333

Предлагается новая релятивистская волновая функция Кулона. Эта функция дается в мультипольном развитии, где удерживаются только выражения необходимые для разрешенных и однократно воспрещенных переходов β. Ввиду зависимости *explicité* от асимптотического момента $\vec{p}$ и спина σ, эта функция позволяет получить поляризацию, угловые корреляции и т. п. непосредственно из соответствующих матричных элементов, равно как полную вероятность перехода. Во всех этих случаях кулонэффект принимается во внимание точным образом.

Я. ЖЕВУСКИЙ, ДВЕ ТЕОРЕМЫ ОБ УРАВНЕНИЯХ ПОЛЯ В СПИНОРНОМ ПРОСТРАНСТВЕ . . . . стр. 335—341

Автором доказаны две теоремы, устанавливающие связь между уравнениями поля первого и второго порядка в спинорном пространстве и уравнениями первого и второго порядка в пространстве Минковского.

З. ШИМАНСКИЙ, ВОЗБУЖДЕННЫЕ УРОВНИ ЯДЕР СО СМЕШАННЫМИ КОНФИГУРАЦИЯМИ. I. МЕТОД ВЫЧИСЛЕНИЙ . . стр. 343—348

Целью работы — исследовать влияние смешивания нуклоновых конфигураций на свойства сферических ядер из промежуточной области ядерных масс $(30 \leqslant A \leqslant 130)$. В этой области наружные протоны и нейтроны находятся на различных оболочках и вследствие их взаимодействия наступает смешивание конфигураций оболочной модели.

Исходной точкой принято считать в первом приближении оболочную модель в $j - j$ связи. Во втором приближении принимаются во внимание остаточные взаимодействия между каждой отдельной парой нуклонов из обеих незамкнутых оболочек. Матричные элементы оператора остаточных взаимодействий были вычислены по стандартному методу в виде линейных комбинаций

интегралов Слейтера. Применялась соответственная пересвязка векторов момента количества движения при использовании символов $9j$ Вигнера. Ввиду того, что недиагональные матричные элементы в общем меньше, чем диагональные, сохраняется грубо приближенная классификация схемы возбужденных уровней согласно квантовым числам сенниорити нейтронов и протонов.

З. ФАЙКЛЕВИЧ, КРАКОВИАНОВЫЙ МЕТОД ОПРЕДЕЛЕНИЯ РЕГИОНАЛЬНОГО ПОЛЯ СИЛЫ ТЯЖЕСТИ . . . . стр. 349–354

В настоящей работе автор представляет метод получения регионального поля силы тяжести на основании исчисленных значений $\Delta g \cdot C$ этой целью автор применяет метод наименьших квадратов способом краковианов. Применение краковианов в данном случае дает возможность быстро получить окончательный результат. В противоположность методам С. М. Симпсона (4) и Ц. Г. Г. Ольдгама, Д В. Сузерлянда нет здесь необходимости применения электронных машин, работа которых в представленном методе уравновешивается промежутком времени около 20 мин, при знании инверса краковиана коэффициентов из уравнения (4).

В настоящей работе дается инверс (11), вычисленный для сети квадратов, растянутой на прямоугольнике $P = 20$ и $Q = 10$ (рис. 1).

В настоящее время разрабатывается каталог инверсов для шести разных размеров сети, обязывающих при предположении, что региональный фон может быть представлен в виде системы кривых полинома второй степени. Элементы краковиана свободных членов L могут быть быстро вычислены при использовании формул (6) и (7). Неизвестные коэффициенты $a_{20}, a_{02} \ldots$ можно быстро получить согласно уравнению (5). Формула (12) служит для контроля полученных значений коэффициентов. В работе, кроме теории, приводится пример применения этого метода на гравиметрической аномалии в окрестностях г. Познани (рис. 2–5).

TABLE DES MATIÈRES

Cena zł 20.—